NASA TECHNICAL NOTE

NASA TN D-2129

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NASA TN D-2129

SECONDARY ERRORS AND OFF-DESIGN CONDITIONS IN OPTIMAL ESTIMATION OF SPACE VEHICLE TRAJECTORIES

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JANUARY 1964

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For sale by the Office of Technical Services, Department of Commerce,
Washington, D.C. 20230 -- Price $1.75

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SUMMARY

To estimate a space vehicle's trajectory by means of linear filter theory, the navigation system and its inputs must be represented mathematically. Inevitably, the representation, or model, is imperfect either because of a deliberate omission or an approximation of certain aspects of the problem for the sake of simplicity, or because of inaccurate a priori knowledge of the system and its inputs. The performance of the system is degraded by such imperfections or, more cogently, is not truly optimum.

For the estimation system to be optimum, all error sources, including those of secondary importance, must be represented in the mathematical model. This generally means that every error affecting system performance is regarded as a state of the system and in an optimal system must be estimated along with the other state variables. Astrodynamic constant uncertainties and bias-type observation errors are used to illustrate this principle and the technique by which errors are represented as state variables. Numerical results are given which show the effect of these two types of errors on the performance of a circumlunar midcourse guidance system.

When the system is not optimum because of an incorrect model, the off-design performance must be analyzed. Equations are developed which can be adjoined to the system equations for computing the performance of the nonoptimal system, using as an example a system with uncompensated bias-type observation errors mixed with the random errors for which the system is designed. Numerical results are given which show that performance can decline substantially in this situation.

INTRODUCTION

In the circumlunar midcourse guidance study reported in references 1 and 2, linear filter theory was used to design an optimal system for estimating a space vehicle's trajectory from a sequence of on-board observations of space angles. An integral part of this optimal system is a mathematical model consisting of the equations of motion assumed for the vehicle in space and the errors entering the system.

The performance figures quoted in the previous studies are legitimate only if the model assumed is valid. This is because in the optimal processing of observational data, the estimation system carries on a running computation of the second-order statistics (covariance matrix) of estimation error. These computed statistics, which depend on the model carried within the system, are interpreted as the performance of the system. Obviously, if the actual system and its inputs are different from the assumed model, the indicated performance figures will differ from the true performance. Therefore, there exists a natural concern regarding the appropriateness of the model, and it is the purpose of this paper to explore this problem area.

In the construction of a model for the earlier studies, only the primary error sources were included because errors from other sources were presumed to be negligible. Besides this omission of small errors there is also the problem of the proper representation of major errors, the description of which always involves an implicit discrepancy because, in practice, the character of the true errors is never known precisely. Even if it were, a deliberate approximation would usually be employed to keep the system simple.

Definitive answers to all possible questions regarding model inaccuracies of the type described above will not be attempted in this report, primarily because the subject is too extensive and, in many areas, somewhat ill-defined. Rather, the emphasis here is on the description of techniques for analyzing any error situation desired. Two examples of model inaccuracies will suffice to illustrate the techniques. These are described in the following paragraphs.

The first inaccuracy to be considered arises from the fact that the equations of vehicle motion within the system are not a perfect representation of the true trajectory dynamics. This gives rise to unavoidable inaccuracies which constitute one of the secondary types of error neglected in the earlier analysis. This is quite a complex error source which will not be thoroughly analyzed here, but one type of computational error which is well defined is convenient to use as an example. This is the inaccuracy due to the uncertainties in the knowledge of the astrodynamic constants which appear in the equations of motion. In this paper it is shown how these uncertainties can be accounted for in the system synthesis and thus optimally compensated in the estimation process. An especially interesting feature of this study is that the system so designed is capable of producing improved estimates of the astrodynamic constants. Although obtaining such estimates is not a design objective for the lunar mission as conceived for these studies, this could be an objective for other space missions, and the results give some preliminary ideas on what might be expected in such an application.

The second type of inaccuracy to be considered has to do with the model assumed for observation errors. In the previous work, results were given only for random observation errors uncorrelated from one observation to the next. The assumption of this dynamical character for the error may not be valid, and it is of interest to investigate the influence which such dynamical misrepresentation may have on system performance. In this report correlated errors are studied by assuming a combination of bias-type[1] and random errors, a model which is artificial but avoids the problem of identifying specific observation instrumentation.

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[1]Bias-type errors are defined as errors which remain constant (or change only according to deterministic laws) over relatively long periods of time.

In the previous work the basic mathematical development for treating any type additive observation error was given and only minor additional detail is required in this report to illustrate how the system is designed for optimal compensation of the assumed bias-plus-random error.

The case of bias-type errors is used further in the report to illustrate off-design analysis. The situation assumed is one in which both bias and random errors are present but the proper compensation for the bias error is not implemented in the system. Two system design possibilities are considered: (1) a complete omission of bias compensation, and (2) a partial compensation of bias. A system so designed is not optimal, so that the covariance matrix of estimation error computed within the system is not a true measure of the performance, but rather only an indication of the performance the system "believes" it achieves. Additional equations are, therefore, necessary to give the true, or off-design, performance.

SYMBOLS

A_i	submatrix of prediction matrix $\Phi^*(t_e,t)$
B	covariance matrix $E\tilde{x}n_2^T$
D	covariance matrix $E\tilde{x}\hat{x}^T$
e	astrodynamic constant uncertainty vector
E	astrodynamic constant gradient matrix
F	perturbation matrix
g_E	acceleration of gravity at Earth's surface
G	gravity gradient matrix
H	submatrix in M relating y to x
I	identity matrix
J	second harmonic of Earth oblateness
K,K*	weighting matrix
M	matrix relating y to x*
n_1	uncorrelated observation error vector
n_2	bias observation error vector
P,P*	covariance matrix of estimation error vector

P'	indicated covariance matrix of estimation error
δP	difference $P-P'$
Q	covariance matrix of n
r	position deviation from reference
$\tilde{r}$	error in estimate of r
$\hat{r}_e$	predicted end-point miss
$\bar{R}$	position vector (subscripts indicate origin and end)
R	magnitude of vector $\bar{R}$
R_E	Earth radius
R_M	Moon radius
S	covariance matrix of velocity correction error
t,τ	general time agreements
t_e	end-point time
t_k	time of kth observation
u	input random vector
U	covariance matrix of u
v	velocity deviation from reference
$\tilde{v}$	error in estimate of v
v_G	velocity to be gained
V	covariance matrix of estimate $\hat{x}$
ΔV	total midcourse velocity correction
W	covariance matrix of x^*
x	six vector of position and velocity deviations from a reference trajectory
x^*	augmented state vector
$\hat{x},\hat{x}^*$	estimates of x,x^*

$\tilde{x},\tilde{x}^*$ error in estimate, $x-\hat{x}$, $x^*-\hat{x}^*$

X,Y,Z space position coordinates

y observation vector

α declination of observed body

β right ascension of observed body

γ half-subtended angle of observed body

Δ increment

μ gravitational constant

σ standard deviation of subscript random variable

Φ,Φ^* transition matrix

Notation Conventions

$(\dot{\ })$, $(\ddot{\ })$ first and second time deriatives of ( )

$(\ )^T$ transpose of matrix ( )

$(\ )^{-1}$ inverse of matrix ( )

$E(\ )$ expected value of ( )

Subscripts

e end point

E Earth

k based on first k observations

M Moon

S Sun

V vehicle

ANALYSIS

Compensation for Astrodynamic Constant Uncertainties

Statement of the problem.- In references 1 and 2 it was assumed implicitly that the only errors in the circumlunar midcourse guidance problem were the errors in injection and in observations. Although these are probably the major sources of error, it is obvious that other errors exist which may become important when the optimal estimation system does a good job in reducing the uncertainties in the knowledge of the trajectory. In this section we will discuss one such additional error, namely, imperfect knowledge of the astrodynamic constants which appear in the equations of motion employed in the optimal filter, and show how this type of error may be optimally compensated.

In the earlier circumlunar navigation study four-body equations of motion were used; that is, the gravitational attractions between the vehicle, Earth, Moon and Sun were taken into account. Also, the second harmonic of Earth's oblateness was used. The astrodynamic constants in these equations are then the gravitational constants of the three celestial bodies, the distances from Earth to Moon and Sun, the Earth oblateness term, and the radius of Earth, a total of seven constants.

Errors in knowledge of all of these constants are not expected to be of equal importance. For instance, the radius of Earth appears only in conjunction with Earth oblateness which already has a small effect. Therefore, we begin by removing Earth radius from consideration.

To evaluate the importance of the remaining six constants, we take the partial derivatives of the equations of motion with respect to these constants. The first equation (eq. (A1) in ref. 1, written in slightly different notation) suffices to illustrate the procedure:

$$\ddot{X}_{EV} = -\frac{\mu_E X_{EV}}{R_{EV}^3}\left[1 + J\left(\frac{R_E}{R_{EV}}\right)^3\left(1 - 5\,\frac{Z_{EV}^2}{R_{EV}^2}\right)\right]$$

$$- \mu_M\left(\frac{X_{MV}}{R_{MV}^3} + \frac{X_{EM}}{R_{EM}^3}\right) - \mu_S\left(\frac{X_{SV}}{R_{SV}^3} + \frac{X_{ES}}{R_{ES}^3}\right) \qquad (1)$$

Taking the derivatives and discarding small terms, we obtain:

$$\Delta\ddot{X}_{EV} = -\left(\frac{X_{EV}}{R_{EV}^3}\right)\Delta\mu_E - \left(\frac{X_{MV}}{R_{MV}^3} + \frac{X_{EM}}{R_{EM}^3}\right)\Delta\mu_M$$

$$-\left(\frac{X_{SV}}{R_{SV}^3} + \frac{X_{ES}}{R_{ES}^3}\right)\Delta\mu_S - \frac{\mu_E X_{EV}}{R_{EV}^3}\left(\frac{R_E}{R_{EV}}\right)^3 \Delta J$$

$$-\mu_M\left[\frac{-3X_{MV}\bar{R}_{MV}\cdot\bar{R}_{EM}}{R_{EM}R_{MV}^5} + \frac{X_{EM}}{R_{EM}}\left(\frac{1}{R_{MV}^3} + \frac{2}{R_{EM}^3}\right)\right]\Delta R_{EM}$$

$$-\mu_S\left[\frac{-3X_{SV}\bar{R}_{SV}\cdot\bar{R}_{ES}}{R_{SV}^5 R_{ES}} + \frac{X_{ES}}{R_{ES}}\left(\frac{1}{R_{SV}^3} + \frac{2}{R_{ES}^3}\right)\right]\Delta R_{ES} \qquad (2)$$

The coefficients of this equation give the sensitivities of the first equation of motion to the various astrodynamic constant errors.

We must next establish the size of the various errors to be assumed. In this study we have used values taken from reference 3 which seem as good as any available, being neither too optimistic nor too pessimistic. It should be noted, however, that the experts do not agree on these values, and furthermore, that with new experiments investigators are constantly seeking to obtain better values for the constants.

The magnitudes of the errors assumed in this study are shown in the second column of table I. Percentage values are given in the third column. These values, when inserted in equation (2), give a measure of the errors in the computed acceleration produced by the astrodynamic constant uncertainties. The acceleration errors are proportional to the coefficients in equation (2), which are functions of the vehicle's position. By evaluating these coefficients at those points in cislunar space where they take on their maximum values, the maximum computed acceleration errors may be determined. These are shown in the last column of table I. It is seen that $\Delta\mu_E$, $\Delta\mu_M$, ΔR_{EM} contribute the major errors, and the study is therefore restricted to these three.

The errors $\Delta\mu_E$, $\Delta\mu_M$, ΔR_{EM}, must be represented in statistical terms for inclusion in the problem. Specifically, a covariance matrix of the errors is required. Since it is not known what statistical parameter the figures in table I are supposed to represent, we assume arbitrarily that these are standard deviations. This might be a pessimistic view since it is possible that the $\pm$ values in table I are actually extremes and therefore should more reasonably correspond to 2 or 3σ values. However, without more detailed knowledge of the manner in which the figures were computed, we will remain on the conservative side.

Correlation between the errors is assumed zero so that the covariance matrix elements are all zero except on the principal diagonal. The justification for this assumption is that the three constants have been determined by essentially independent experiments. The most accurate value of μ_E has been obtained from measurements of Earth's radius and the acceleration of gravity on Earth's surface:

$$\mu_E = R_E^2 g_E \tag{3}$$

The Moon's gravitational constant is computed from μ_E and the ratio of Moon and Earth masses:

$$\mu_M = \frac{M_M}{M_E} \mu_E \tag{4}$$

where the mass ratio has been computed from experiments entirely unrelated to those used to compute μ_E. Since the mass ratio is far less accurately known than is μ_E, most of the error in μ_M is due to the mass ratio error, and, for all practical purposes, $\Delta\mu_M$ is independent of $\Delta\mu_E$.

The best figures available for R_{EM}, the Earth-Moon distance, have been obtained by bouncing radio signals off the Moon. A knowledge of the radii of Earth and Moon are required to compute R_{EM} from this data, which implies a correlation between $\Delta\mu_E$ and ΔR_{EM}. However, the uncertainties in the knowledge of these radii are far less than that in the radar measurement itself. Thus, ΔR_{EM} is an essentially independent random variable.

The estimation equations.- To design an estimation system, we seek to utilize the solution to the estimation problem given in references 1 and 4. This requires that the equations of the stochastic processes involved in estimation be represented in the linear form for which the optimal estimation solution has been obtained:

$$\dot{x}^*(t) = F(t)x^*(t) + u^*(t) \tag{5}$$

$$y(t) = M(t)x^*(t) \tag{6}$$

Here, x^* is the "state" of the system, u^* is a "white noise" vector-valued stochastic input process, and y is the observation process, or output of the system.

By examining equation (6) one can see that all quantities which affect the system output, y, must be represented in the state vector, x^*. Since observations in general are functions of the vehicle position and the observation errors, equations for both of these processes are required.

Consider first the vehicle motion equations. For a system which optimally compensates for the astrodynamic constant uncertainties, the astrodynamic constants must be regarded as variables in the same manner as are the components of vehicle position and velocity; that is, the equations of motion are written as functions of $X, Y, Z, \mu_E, \mu_M,$ and R_{EM}:

$$\ddot{X} = f_1(X, Y, Z, \mu_E, \mu_M, R_{EM}) \tag{7a}$$

$$\ddot{Y} = f_2(X, Y, Z, \mu_E, \mu_M, R_{EM}) \tag{7b}$$

$$\ddot{Z} = f_3(X, Y, Z, \mu_E, \mu_M, R_{EM}) \tag{7c}$$

These equations must be linearized for representation in the form (5). The procedure is an expansion in Taylor series discarding all but the first-order terms. For instance, for equation (7a) we obtain

$$\ddot{X} + \Delta\ddot{X} = f_1 + \frac{\partial f_1}{\partial X}\Delta X + \frac{\partial f_1}{\partial Y}\Delta Y + \frac{\partial f_1}{\partial Z}\Delta Z$$

$$+ \frac{\partial f_1}{\partial \mu_E}\Delta\mu_E + \frac{\partial f_1}{\partial \mu_M}\Delta\mu_M + \frac{\partial f_1}{\partial R_{EM}}\Delta R_{EM} \tag{8}$$

where f_1 and its partials are evaluated for a reference trajectory corresponding to the nominal, or mean, values of $X, Y, Z,$ and the astrodynamic constants. The quantities $\Delta X, \ldots, \Delta R_{EM}$ are then random variables with zero mean. Equation (8) can be rewritten

$$\Delta\ddot{X} = \frac{\partial f_1}{\partial X}\Delta X + \frac{\partial f_1}{\partial Y}\Delta Y + \frac{\partial f_1}{\partial Z}\Delta Z$$

$$\frac{\partial f_1}{\partial \mu_E}\Delta\mu_E + \frac{\partial f_1}{\partial \mu_M}\Delta\mu_M + \frac{\partial f_1}{\partial R_{EM}}\Delta R_{EM} \tag{9}$$

If the vector x is defined as

$$x = \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \\ \Delta \dot{X} \\ \Delta \dot{Y} \\ \Delta \dot{Z} \end{Bmatrix} \tag{10}$$

and the vector e as

$$e = \begin{Bmatrix} \Delta\mu_E \\ \Delta\mu_M \\ \Delta R_{EM} \end{Bmatrix} \tag{11}$$

it is seen that equations (7) can be written in the linearized form

$$\dot{x} = \begin{bmatrix} F_X & E \end{bmatrix} \begin{Bmatrix} x \\ e \end{Bmatrix} \tag{12}$$

where the F_X matrix contains the partials of f_1, f_2, f_3 with respect to X, Y, and Z, and E the partials of f_1, f_2, f_3 with respect to the astrodynamic constants. Thus, equation (12) is in the required form. Note that there is no input, u, in equation (12) because the vehicle is assumed in a free-fall condition.

Considering now the observation error equations, we note that observations in general are of the form

$$y(t) = H(t)x(t) + n(t) \tag{13}$$

where H is a matrix of partials of those quantities being observed with respect to the components of x (i.e., vehicle position and velocity), and n(t) is the additive error. It will be assumed here that the error n(t) can be represented by the linear equations

$$\dot{w}(t) = F_W(t)w(t) + u(t) \tag{14}$$

$$n(t) = \Gamma(t)w(t) \tag{15}$$

Here, w(t) is an assumed basic underlying error process with white noise input u(t) and dynamics $F_W(t)$, and Γ represents the manner in which components of w are combined to form n. Thus, equation (13) can be written

$$y(t) = H(t)x(t) + \Gamma(t)w(t) \tag{16}$$

which is of the form (6) if w and x are parts of the state vector x^*. By adjoining equation (14) to equation (12) we have an equation of the form (5).

The system state vector is now defined as

$$x^* = \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \\ \Delta\dot{X} \\ \Delta\dot{Y} \\ \Delta\dot{Z} \\ \Delta\mu_E \\ \Delta\mu_M \\ \Delta R_{EM} \\ w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_p \end{Bmatrix} = \begin{Bmatrix} x \\ e \\ w \end{Bmatrix} \tag{17}$$

where x is the six vector of position and velocity deviations from nominal, e is the three vector of astrodynamic constant uncertainties, and w is the p vector of error sources contributing to the observation error. The equations, or mathematical model, of the system can now be written in partitioned form as

$$\begin{Bmatrix} \dot{x} \\ \dot{e} \\ \dot{w} \end{Bmatrix} = \begin{bmatrix} F_x & E & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F_w \end{bmatrix} \begin{Bmatrix} x \\ e \\ w \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ u \end{Bmatrix} \tag{18}$$

$$\{y\} = \begin{bmatrix} H & 0 & \Gamma \end{bmatrix} \begin{Bmatrix} x \\ e \\ w \end{Bmatrix} \tag{19}$$

The square matrix on the right in equation (18) is identified as the F matrix of equation (5), and the matrix in equation (19) is the M of equation (6). The middle submatrix in the second row of F is null because there are no dynamics associated with e; that is, $\dot{e} = 0$, or $e(t) = e(t_o)$. The other null submatrices in F are due to the assumption that the dynamics of x, e, and w are uncoupled. The null submatrix in (19) is null because the observations do not depend directly upon e.

The submatrices F_x and E in (18) are made up of partial derivatives as described earlier:

$$F_x = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} & 0 & 0 & 0 \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} & 0 & 0 & 0 \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial f_1}{\partial \mu_E} & \frac{\partial f_1}{\partial \mu_M} & \frac{\partial f_1}{\partial R_{EM}} \\ \frac{\partial f_2}{\partial \mu_E} & \frac{\partial f_2}{\partial \mu_M} & \frac{\partial f_2}{\partial R_{EM}} \\ \frac{\partial f_3}{\partial \mu_E} & \frac{\partial f_3}{\partial \mu_M} & \frac{\partial f_3}{\partial R_{EM}} \end{bmatrix} \tag{21}$$

The solution of equation (18) is:

$$x^*(t) = \Phi(t,t_o)x^*(t_o) + \int_{t_o}^{t} \Phi(t,\tau)u^*(\tau)d\tau \tag{22}$$

where $\Phi(t,t_o)$ is the "transition" matrix of the system and is the solution of the matrix differential equation

$$\dot{\Phi}(t,t_o) = F(t)\Phi(t,t_o) \tag{23}$$

with initial conditions $\Phi(t_o,t_o) = I$.

An optimal[2] estimate of the vehicle's position and velocity for use in guiding the vehicle can be obtained, as shown in references 1 (eq. (9)) and 4 (eq. (21)) by processing the sequence of observations by means of the linear estimation equations:

$$\hat{x}^*_{k-1}(t_k) = \Phi^*(t_k,t_{k-1})\hat{x}^*_{k-1}(t_{k-1}) \tag{24}$$

$$\hat{x}^*_k(t_k) = \hat{x}^*_{k-1}(t_k) + K(t_k)[y(t_k) - M(t_k)\hat{x}^*_{k-1}(t_k)] \tag{25}$$

where $\hat{x}^*$ is the estimate of x^*, t_k is the time of the kth observation in the sequence, M is a matrix of partials of the observation variables with respect to the state variables, and K is the optimal weighting matrix. The subscripts k-1 and k on $\hat{x}^*$ are used here to indicate that the estimate is based respectively, on the first k-1 and k of the sequence of observations.

The weighting matrix K for use in equation (25) is computed from the following equations (equivalent to 6, 10, 11 in ref. 1, and eqs. (28), (29), (30), (32) of ref. 4)

$$U'(t_k,t_{k-1}) = \int_{t_{k-1}}^{t_k} \Phi^*(t_k,\tau)U(\tau)\Phi^{*T}(t_k,\tau)d\tau \tag{26}$$

$$P_{k-1}(t_k) = \Phi^*(t_k,t_{k-1})P_{k-1}(t_{k-1})\Phi^{*T}(t_k,t_{k-1}) + U'(t_k,t_{k-1}) \tag{27}$$

$$K(t_k) = P_{k-1}(t_k)M^T(t_k)[M(t_k)P_{k-1}(t_k)M^T(t_k)]^{-1} \tag{28}$$

$$P_k(t_k) = [I - K(t_k)M(t_k)]P_{k-1}(t_k) \tag{29}$$

where P_k is the covariance matrix of the error in estimate based on k observations:

$$P_k = E\tilde{x}_k{}^*\tilde{x}^*{}_k{}^T \tag{30}$$

$$\tilde{x}^*_k = x^*-\hat{x}^*_k \tag{31}$$

and U is the covariance matrix of the input, u:

$$U = Euu^T \tag{32}$$

---

[2]Optimal estimate is here defined as the minimum variance unbiased estimate. See reference 4.

It is noted that the vector estimate obtained from the above computations includes estimates of the observation errors and of the astrodynamic constant uncertainties. In the guidance problem these are of no interest in themselves but are required if the other state variables, vehicle position and velocity, are to be estimated optimally.

If the observation errors are random (i.e., uncorrelated from one observation to the next), the estimate of this portion of the state vector is zero just prior to processing an observation, hence, of no use in estimating the other state variables and may be omitted, with a resultant simplification of the system as shown in the appendix. This is the assumption which will be made here to avoid unnecessary complexity. Under this assumption the state vector may be redefined as

$$x^* = \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \\ \Delta \dot{X} \\ \Delta \dot{Y} \\ \Delta \dot{Z} \\ \Delta \mu_E \\ \Delta \mu_M \\ \Delta R_{EM} \end{Bmatrix} = \begin{Bmatrix} x \\ e \end{Bmatrix} \tag{33}$$

With Φ^* also suitably redefined for the reduced system, the system equation becomes

$$x^*(t) = \Phi^*(t,t_o)x^*(t_o) \tag{34}$$

where there is now no input or forcing function. The estimation equations become

$$\hat{x}^*_{k-1}(t_k) = \Phi^*(t_k,t_{k-1})\hat{x}^*_{k-1}(t_{k-1}) \tag{35}$$

$$x^*_{k-1}(t_k) = x^*_{k-1}(t_k) + K(t_k)[y(t_k) - H(t_k)\hat{x}^*_{k-1}(t_k)] \tag{36}$$

where H is the portion of M not related to observation errors. The weighting matrix and variance equations are now

$$P_{k-1}(t_k) = \Phi^*(t_k,t_{k-1})\ P_{k-1}(t_{k-1})\ \Phi^{*T}(t_k,t_{k-1}) \tag{37}$$

$$K(t_k) = P_{k-1}(t_k)\ H^T(t_k)[H(t_k)\ P_{k-1}(t_k)\ H^T(t_k) + Q(t_k)]^{-1} \tag{38}$$

$$P_k(t_k) = [I - K(t_k)\ H(t_k)]\ P_{k-1}(t_k) \tag{39}$$

where it is noted that the effect of the forcing function $u(t)$ does not appear in the variance equation, (37), as it did before, but must appear as Q in equation (38) as shown.

The H matrix is defined by the relation

$$y = Hx^* + n \tag{40}$$

where $n = \Gamma w$ is the additive observation error. If we assume observations containing three components α, β, and γ, the declination, right ascension, and subtended angles of the observed planet (as in ref. 1), illustrated in figure 1, the H matrix is a 3×9 as follows:

$$H = \begin{bmatrix} \frac{\partial\alpha}{\partial x} & \frac{\partial\alpha}{\partial y} & \frac{\partial\alpha}{\partial z} & 0 & 0 & 0 & 0 & 0 & \frac{\partial\alpha}{\partial R_{EM}} \\ \frac{\partial\beta}{\partial x} & \frac{\partial\beta}{\partial y} & \frac{\partial\beta}{\partial z} & 0 & 0 & 0 & 0 & 0 & \frac{\partial\beta}{\partial R_{EM}} \\ \frac{\partial\gamma}{\partial x} & \frac{\partial\gamma}{\partial y} & \frac{\partial\gamma}{\partial z} & 0 & 0 & 0 & 0 & 0 & \frac{\partial\gamma}{\partial R_{EM}} \end{bmatrix} \tag{41}$$

The zeros in H mean that the α, β, γ do not depend on the vehicle velocity or upon μ_E and μ_M. Also, if the observation is of the Earth, the angles do not depend upon R_{EM}. However, the Moon angles α_M, β_M, γ_M do depend on R_{EM}. Analytic expressions for the required partials are given in table II.

For the computation of equation (37) the transition matrix is required. This differs from the transition matrix used in references 1 and 2 by inclusion of the effects of the astrodynamic constant uncertainties. The development is parallel to that given in reference 1. The linearized system equation (34), written in the differential equation form, is

$$\dot{x}^* = Fx^* \tag{42}$$

which, partitioned into position, velocity, and astrodynamic constant uncertainty portions, is

$$\begin{Bmatrix} \dot{r} \\ \dot{v} \\ \dot{e} \end{Bmatrix} = \begin{bmatrix} 0 & I & 0 \\ G & 0 & E \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} r \\ v \\ e \end{Bmatrix} \tag{43}$$

The null matrices in the bottom row mean that e is constant (i.e., $\dot{e} = 0$). The G matrix may be termed the "gravity gradient" matrix, which consists of the partials of the gravitational forces acting upon the vehicle with respect to the vehicle position components. Similarly, the E matrix consists of the partials of the gravitational forces with respect to the astrodynamic constants. Analytic expressions for these partials are readily computed from the equations of motion.

The transition matrix can now be obtained as the solution of the matrix differential equation

$$\dot{\Phi}^* = F\Phi^* \tag{44}$$

where

$$F = \begin{bmatrix} 0 & I & 0 \\ G & 0 & E \\ 0 & 0 & 0 \end{bmatrix}$$

Initial conditions $\Phi^*(t_{k-1},t_{k-1}) = I$ are introduced at time t_{k-1} and equation (44) is integrated until time t_k to obtain $\Phi^*(t_k,t_{k-1})$ for use in equation (37).[3]

The guidance equation.- As in the case of the estimation computation, the guidance computations will be different from those given in reference 2 to include the effects of the new state variables. As before, we presume the use of a fixed-time-of arrival guidance law, in which the estimated end-point miss is computed and then a velocity correction computed to null this miss. The end-point miss estimate is given by

$$\hat{x}^*(t_e) = \Phi^*(t_e,t)\hat{x}^*(t) \tag{45}$$

where in partitioned form $\Phi^*(t_e,t)$ may be written

$$\Phi^*(t_e,t) = \left[\begin{array}{cc|c} A_1 & A_2 & A_5 \\ A_3 & A_4 & A_6 \\ \hline 0 & 0 & I \end{array}\right] \tag{46}$$

Each A_i submatrix is a (3×3). The upper left (6×6) is the original prediction matrix of reference 2, and the added rows and columns result from the astrodynamic constant uncertainties. The A_5 is the matrix which relates position deviations at the end to errors in the astrodynamic constants, and A_6 similarly gives the velocity deviations. The null matrices mean that the e vector is not affected by position and velocity deviations, and the identity matrix means simply that e is constant.

[3]It may be noted that, alternatively, equation (37) may be solved for P by integrating the differential equation

$$\dot{P} = FP + PF^T$$

which does not require Φ^* at all. However, in our application Φ^* is used as part of the guidance law and its separate computation is expedient.

Now, if it is desired to null the indicated end-point miss, given by

$$\hat{r}_e = [A_1 \quad A_2 \quad A_5]\hat{x}^* \tag{47}$$

we must add a velocity increment v_G such that

$$\hat{r}_e + A_2 v_G = 0 \tag{48}$$

where $A_2 v_G$ gives the effect on the end-point position of a velocity change v_G at the present time. From equations (47) and (48) we then have

$$v_G = -A_2^{-1} [A_1 \quad A_2 \quad A_5]\hat{x}^* \tag{49}$$

which is exactly the same as given in reference 2 except for the addition of A_5 and e. In other words there has been added to v_G the term $-A_2^{-1} A_5 \hat{e}$ which is the velocity correction necessary to null the miss implied by the estimate $\hat{e}$. Of course, if the estimated errors in the astrodynamic constants are quite small (as they usually would be), the added term has negligible effect on the computations. However, it is included here for completeness.

Guidance system performance statistics.- Assessment of the statistical performance of a complete guidance system generally requires the computation of the covariance matrices of each of the three random variables x, $\hat{x}$, and $\tilde{x}$. Each of these is in some sense a measure of the performance of the system. The quantity x is the deviation from nominal and thus at the end point can be interpreted as the "miss" experienced. The estimate, $\hat{x}$, is used to compute midcourse maneuvers, hence, is a measure of the fuel used in these maneuvers. The estimation error, $\tilde{x}$, which exists at completion of the midcourse task is the initial uncertainty with which the subsequent guidance mode (e.g., atmospheric entry guidance) must cope.

Equations for computing the covariance matrix of $\tilde{x}$ have already been given in the development of the estimation system. We need to add to these the equations required to compute the covariance matrices of x and $\hat{x}$. These will differ slightly from the versions given in reference 2 to take into account the effects of the uncertainty e.

For periods between velocity corrections, the deviation covariance matrix is obtained by computing

$$\begin{aligned} W(t) &= Ex^*(t)x^{*T}(t) \\ &= \Phi^*(t,t_o)W(t_o)\Phi^{*T}(t,t_o) \end{aligned} \tag{50}$$

(The covariance matrix of x is available as a submatrix in W.) When a velocity correction is made, a step change in W occurs. The required equation, derived using the methods of reference 2 (appendix D), is

$$W_c = C(W - P)C^T + P + S \tag{51}$$

where W_c is the covariance matrix after the correction, S is the covariance matrix of correction error, and

$$C = \begin{bmatrix} I & 0 & 0 \\ -A_2^{-1}A_1 & 0 & -A_2^{-1}A_5 \\ 0 & 0 & I \end{bmatrix} \tag{52}$$

written in partitioned form.

The covariance matrix of the estimate is obtained in the following manner. Since the estimate and error in estimate are orthogonal (i.e., $E\hat{x}\tilde{x}^T = 0$) for an optimal estimate (see ref. 4), we have, from the relation $x = \hat{x} + \tilde{x}$,

$$Exx^T = E\hat{x}\hat{x}^T + E\tilde{x}\tilde{x}^T \tag{53}$$

or

$$E\hat{x}\hat{x}^T \equiv V = W - P \tag{54}$$

Thus, with P and W as computed from the previously developed equations, V is obtained directly by use of equation (54).

Compensation for Mixed Bias and Random Observation Errors

In this section a more complex model will be assumed for the observation error than has heretofore been considered. The astrodynamic constant uncertainties will be ignored - that is, assumed zero.

Suppose that the observation error consists of two additive components, one random (uncorrelated from one observation to the next) and the other a bias; that is, the equation for the observations is of the form[4] (compare eq. (13))

$$y(t) = H(t)x(t) + n_1(t) + n_2(t) \tag{55}$$

[4]Observations are considered to be three-component vectors as in reference 1, but the analysis is easily extended to any other situation desired.

where

n_1 a random error with covariance matrix Q_1

n_2 a bias error with covariance matrix Q_2

H a matrix of partial derivatives of observables with respect to state variables

The differential equations of the n_1 and n_2 processes are assumed to be given by

$$\dot{n}_1(t) = F_1(t)n(t) + u_1(t) \tag{56}$$

$$\dot{n}_2(t) = 0 \tag{57}$$

The solutions of equations (56) and (57) are:

$$n_1(t) = \Phi_1(t,t_o)n_1(t_o) + \int_{t_o}^{t} \Phi_1(t,\tau)u_1(\tau)d\tau \tag{58}$$

$$n_2(t) = n_2(t_o) \tag{59}$$

The assumption that n_1 is uncorrelated from one observation to the next implies that the observations are sufficiently far apart that $\Phi_1(t_{k+1},t_k)$ is for all practical purposes zero, where t_{k+1} and t_k are the times of the $k+1$ and kth observations, respectively.

The state vector for the system as here defined is seen to be

$$x^* = \begin{Bmatrix} x \\ n_1 \\ n_2 \end{Bmatrix} \tag{60}$$

and the system equation in partitioned form is

$$\begin{Bmatrix} x(t) \\ n_1(t) \\ n_2(t) \end{Bmatrix} = \begin{bmatrix} \Phi_x(t,t_o) & 0 & 0 \\ 0 & \Phi_1(t,t_o) & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} x(t_o) \\ n_1(t_o) \\ n_2(t_o) \end{Bmatrix} + \begin{Bmatrix} 0 \\ u_1'(t,t_o) \\ 0 \end{Bmatrix} \tag{61}$$

$$y(t_k) = [H(t_k) \quad I \quad I] \begin{Bmatrix} x(t_k) \\ n_1(t_k) \\ n_2(t_k) \end{Bmatrix} \tag{62}$$

where

$$u_1'(t,t_o) = \int_{t_o}^{t} \Phi_1(t,\tau)u_1(\tau)d\tau$$

The equations for optimal estimation are the same as in the problem treated in the previous section (eqs. (24) through (29)), except that here we use a different state vector and suitably redefined K, P, M, and Φ matrices. These equations could be used "as is." However, the assumption that n_1 is uncorrelated between observations permits the same type of computation contraction described in the previous section - that is, the estimation of n_1 can be omitted. The details of this contraction are given in the appendix.

In addition, it is shown in the appendix how the estimation equations appear in partitioned form. In this form the additional computations required because of the more complex (bias plus random) error model are clearly illustrated. However, it should be noted that implementing the computations in the partitioned form, where all matrix operations are of order 6×6 or less, is not necessarily simpler than using the 9×9 format.

Bias Error Off-Design Performance Analysis

No compensation for bias.- Here we are concerned with off-design performance analysis. The situation selected for illustration in this section is one in which mixed random and bias error exists but the bias error is not accounted for in any way in the estimation procedure. The problem is to determine what performance degradation will result.

In this case it is assumed that, so far as the estimation system knows, only the uncorrelated observation error exists. The estimation equations are then those which are optimal for this type of situation - that is, only the estimate of vehicle position and velocity is formed, and the computed P matrix represents what the system "believes" to be the statistics of the error in estimating position and velocity. The equations used for processing an observation are those which are optimal for an assumed purely random error model (see ref. 1, eqs. (12) through (14)):

$$\hat{x}_k = \hat{x}_{k-1} + K(y - H\hat{x}_{k-1}) \tag{63}$$

$$K = P'_{k-1}H^T(HP'_{k-1}H^T + Q_1)^{-1} \tag{64}$$

$$P'_k = (I - KH)P'_{k-1} \tag{65}$$

and for updating the estimate between observations,

$$\hat{x}(t_k) = \Phi(t_k,t_{k-1})\hat{x}(t_{k-1}) \tag{66}$$

$$P'(t_k) = \Phi(t_k,t_{k-1})P'(t_{k-1})\Phi^T(t_k,t_{k-1}) \tag{67}$$

A prime is used here on P to distinguish between the indicated error covariance matrix, P', and the true P.

In order to obtain second-order statistics of system performance we require, as before, equations for computing the covariance matrices

$$E\tilde{x}\tilde{x}^T = P$$

$$E\hat{x}\hat{x}^T = V$$

$$Exx^T = W$$

The computational equations for W will be the same as those given in the section on astrodynamic constant uncertainties, but the equations for P and V will differ because here the system is nonoptimal. For instance, we note that P is the true covariance matrix of estimation error as opposed to what the system "believes" is its performance - that is, P as computed from equations (65) and (67).

The development of equations for P and V divides naturally into consideration of (1) the computation of the matrices at the times when observations or velocity corrections are made and (2) the computations which occur in intervals between these points.

We begin first with the step changes, and consider what computations are necessary at the observation times. Using the estimation equation (63) and the relations

$$\hat{x}_k = x - \tilde{x}_k \tag{68}$$

$$y = Hx + n_1 + n_2 \tag{69}$$

we obtain an expression for the propagation of estimation error when an observation is processed:

$$\tilde{x}_k = [I - KH]\tilde{x}_{k-1} - K[n_1 + n_2] \tag{70}$$

The argument t_k, the time of the observation, is omitted from this and the following equations because it is the same for all quantities.

A recursion equation for computing P is developed directly from equation (70).

$$\begin{aligned}\tilde{x}_k\tilde{x}_k^T &= (I - KH)\tilde{x}_{k-1}\tilde{x}_{k-1}^T(I - KH)^T + K(n_1 + n_2)(n_1 + n_2)^T K^T \\ &\quad - (I - KH)\tilde{x}_{k-1}(n_1 + n_2)^T K^T - K(n_1 + n_2)\tilde{x}_{k-1}^T(I - KH)^T \end{aligned}\tag{71}$$

Now n_1 is an observation error uncorrelated with previous values of n_1 and therefore uncorrelated with $\tilde{x}_{k-1}$. Also, n_1 and n_2 are assumed uncorrelated with each other. Thus, taking the expected value of equation (71), we obtain:

$$\begin{aligned}P_k &= (I - KH)P_{k-1}(I - KH)^T + K(Q_1 + Q_2)K^T \\ &\quad - (I - KH)B_{k-1}K^T - KB_{k-1}^T(I - KH)^T\end{aligned}\tag{72}$$

where B_{k-1} is defined as the covariance matrix

$$B_{k-1} = E\tilde{x}_{k-1}n_2^T \tag{73}$$

Note that B_k gives the correlation statistics of the error in estimate, $\tilde{x}_{k-1}$, and the bias error, n_2. Obviously, a recursion equation for B_k is also required, which is obtained by multiplying equation (70) by n_2^T and taking the expected value:

$$B_k = (I - KH)B_{k-1} - KQ_2 \tag{74}$$

It is readily seen that if B_k is computed before P_k, a somewhat simpler expression for P_k can be utilized; combining equations (74) and (72), we obtain

$$P_k = (I - KH)P_{k-1}(I - KH)^T + K(Q_1 - Q_2)K^T - B_kK^T - KB_k^T \tag{75}$$

Since we are interested in the difference between the true and indicated P matrices, it is instructive to produce another recursion equation for the computation of this difference matrix, which we might call $\delta P = P - P'$. Using equations (75), (64), and (65), we find quite easily that

$$\delta P_k = (I - KH)\delta P_{k-1}(I - KH)^T - KQ_2K^T - B_kK^T - KB_k^T \tag{76}$$

The equation for computing $V = E\hat{x}\hat{x}^T$ at the time of an observation is developed as follows. Since $x = \hat{x} + \tilde{x}$, we can write

$$Exx^T = E\hat{x}\hat{x}^T + E\hat{x}\tilde{x}^T + E\tilde{x}\hat{x}^T + E\tilde{x}\tilde{x}^T \tag{77}$$

Using the letter designation

$$E\tilde{x}\hat{x}^T = D \tag{78}$$

we then can write

$$E\hat{x}_k\hat{x}_k^T = V_k = (W - P_k) - (D_k + D_k^T) \tag{79}$$

The P_k and W in this expression are obtained from computations of equations given previously for these matrices. For an optimal unbiased estimate as obtained with the system of the previous section, D is zero - that is, $\hat{x}_k$ and $\tilde{x}_k$ are uncorrelated. However, in the present case this is not so, and we must have a recursion formula for computing D. Using equations (63), (70), and (74), it is seen that

$$D_k = (I - KH)(D_{k-1} + P_{k-1} - H^TK^T) + K(Q_1 - Q_2)K^T + B_kK^T + KB_k^T \tag{80}$$

Other formulas are possible, depending upon what is computed first. For instance,

$$D_k = (I - KH)(D_{k-1} + \delta P_{k-1}) - \delta P_k \tag{81}$$

which is useful if δP is being computed (by means of eq. (76)).

Now consider the second part of the development, which has to do with the changes in P and V that occur in intervals between observations. The time-transition equations to be used for updating between observations are easily derived by noting that the estimation error propagates in the same way as does the estimate (eq. (66)):

$$\tilde{x}(t_k) = \Phi(t_k,t_{k-1})\tilde{x}(t_{k-1}) \tag{82}$$

Also, n_2 is constant. Thus, we have

$$P(t_k) = \Phi(t_k,t_{k-1})P(t_{k-1})\Phi^T(t_k,t_{k-1}) \tag{83}$$

$$\delta P(t_k) = \Phi(t_k,t_{k-1})\delta P(t_{k-1})\Phi^T(t_k,t_{k-1}) \tag{84}$$

$$B(t_k) = \Phi(t_k,t_{k-1})B(t_{k-1}) \tag{85}$$

$$D(t_k) = \Phi(t_k,t_{k-1})D(t_{k-1})\Phi^T(t_k,t_{k-1}) \tag{86}$$

With these equations to obtain P and D, and equation (50) and (51) for W, V_k is obtained by use of equation (79).

Taken altogether, the foregoing equations give the effect of processing an observation and of transition between observations, and thus are sufficient for the complete determination of the statistics of system performance. Initial conditions for the various matrices are required, of course. If the problem begins at injection into the translunar trajectory, then $P(t_o)$ may be the covariance matrix of injection errors. The matrix $B(t_o)$ will be zero because the initial estimate is by assumption not dependent upon the bias error, n_2. The $D(t_o)$ matrix will be zero if the initial estimate is optimal; otherwise, some other value must be used.

Partial compensation of bias.- In this section we will consider a design compromise in which the bias error is compensated but not in the optimal fashion described earlier. This might be called a partial compensation. The idea to be developed here is (1) to use the simple no-bias estimation equation

$$\hat{x} = \hat{x}_{k-1} + K[y - H\hat{x}_{k-1}] \tag{87}$$

where no estimate of the bias is formed for estimating x (cf. eq. (25)), and (2) to find the K which is optimum for the use of this equation. The estimate formed in this way is optimum in a sense but is not unbiased. Thus, the rms estimation error should be larger than that of the unbiased estimate.

To find the optimum K we may use a variational technique, proceeding as follows. The recursion relations for the estimation error covariance matrix are given by equations (72) and (83), repeated here for convenience:

$$P_k = (I - KH)P_{k-1}(I - KH)^T + K(Q_1 + Q_2)K^T$$

$$- (I - KH)B_{k-1}K^T - KB_{k-1}^T(I - KH)^T \quad \text{at} \quad t_k \tag{72}$$

$$P_{k-1}(t_k) = \Phi(t_k,t_{k-1})P_{k-1}(t_{k-1})\Phi^T(t_k,t_{k-1}) \tag{83}$$

Following the common procedure, K is chosen to minimize the expected squared error in estimating some linear function of x. (For instance, it may be desired to minimize the mean-square error in the estimate of the end-point miss.) If $z = \Omega x$ is the criterion vector whose mean-square estimation error is to be minimized, we observe that since after k observations

$$\tilde{z} = \Omega\tilde{x}_k \tag{88}$$

we wish to minimize the functional

$$\mathrm{tr}(E\tilde{z}\tilde{z}^T) = \mathrm{tr}(\Omega P_k \Omega^T) \tag{89}$$

Substituting into (89) from equation (72), we then let $K = K_{opt} + \eta K_1$, where K_{opt} is the optimum weighting function, K_1 is an arbitrary matrix of the proper dimensions, and η is a scalar Lagrangian multiplier. Then, differentiating with respect to η, letting η go to zero, and setting the resulting expression equal to zero, we obtain:

$$\mathrm{tr}\ \Big\{ \Omega[K_{opt}(HP_{k-1}H^T + Q_1 + Q_2 + HB_{k-1} + B_{k-1}^T H^T)$$

$$- (P_{k-1}H^T + B_{k-1})]K_1{}^T\Omega^T \Big\} = 0 \tag{90}$$

Since Ω is not zero and K^T_1 is arbitrary, this expression can be zero only when

$$K_{opt} = (P_{k-1}H^T + B_{k-1})(HP_{k-1}H^T + Q_1 + Q_2 + HB_{k-1} + B^T_{k-1}H^T)^{-1} \tag{91}$$

Using the value of K given by (91) in equation (72), we obtain as the recursion equation for computing the change in P when an observation is processed:

$$P_k = P_{k-1} - K(HP_{k-1} + B^T_{k-1}) \quad \text{at} \quad t_k \tag{92}$$

Equations (91) and (92) are seen to be quite analogous to the no-bias optimal equations, the difference being that the covariance matrix B must now be computed. Equations (74) and (85) given in the previous section serve this purpose.

It will be noted now that a system designed in the manner described here is not really much simpler than the true optimal system because of the necessity of computing B. This is seen when equation (91) is compared with equation (A24) and observing that B_{k-1} is defined identically with the P_{xn_2} of equation (A24), and P_{n_2} is equal to Q_2 if the estimate of n_2 is always taken to be zero. Thus, with the computation of B implemented, all the information necessary to form a truly optimal estimate is available and requires only a few extra matrix multiplications and some additional computer storage. For this reason, this type of restricted-optimal system is considered to be of little more than academic interest and its study will not be pursued further.

RESULTS

Compensation for Astrodynamic Constant Uncertainties

A digital computer program was written to simulate the guidance system design described in the Analysis section for optimal compensation of the astrodynamic constant uncertainties, $\Delta\mu_E$, $\Delta\mu_M$, and ΔR_{EM}. Results obtained using this program are reported in this section.

The situation simulated was the same as that employed in references 1 and 2 and may be described as follows. It is assumed that a space vehicle is injected onto a circumlunar trajectory, the rms injection errors being one kilometer in position and one meter per second in velocity along each of the three axes of an earth-centered nonrotating coordinate system. The trajectory passes at a distance of 4766 km above the lunar surface after about 3-1/4 days of flight and returns to a reentry corridor after 6-1/2 days.

Observations made during the flight are assumed to consist of the simultaneous measurement of three angles, the declination (α), the right ascension (β), and the half-subtended angle (γ) of either the Earth or the Moon as seen from the vehicle. The geometry is illustrated in figure 1 for Earth observations. No attempt is made to describe an instrumentation system which could make such measurements, the assumption being simply that the system is subject to random gaussian errors added independently to each of the three angle measurements. The standard deviation of the errors is assumed to be given by the formula

$$\sigma_n = \sqrt{100 + (0.0017\gamma)^2} \text{ sec arc} \tag{93}$$

where γ is the half-subtended angle of the body being observed, expressed in seconds of arc.

Three different schedules of observations and velocity corrections were employed. The first is the schedule described in reference 2 which consists of 426 observations on the outgoing leg of the trip, and three velocity corrections. The measurements alternate, in groups, between observations of the Earth and the Moon. Only the first half (i.e., the outbound portion) of this schedule was used here.

The other two schedules employed are a "short" schedule having only 80 observations during the entire 6-1/2-day flight, and a "long" schedule with a total of 400 observations. Five velocity corrections are assumed. These schedules are shown in figures 2 and 3, which are plots of the nominal trajectory with the times and type of observations indicated. The times of the five velocity corrections are shown as circles on the trajectory. These schedules are the result of a scheduling quasi-optimization, the discussion of which is beyond the scope of this report.

It should be noted that observations on the short schedule are always spaced at least 30 minutes apart, whereas on the long schedule observations are 6 minutes apart during each observation period. There are several fairly long periods of time in each schedule during which no observations are made.

For assessment of the performance of the over-all vehicle control system, it was necessary to simulate random errors in the implementation of the velocity corrections. These were assumed to be represented by a 0.5° rms pointing error for the rocket engine and 0.1 m/sec rms error in the magnitude of the correction. It was assumed that the corrections were monitored with an accuracy of 0.01 m/sec in each of three orthogonal inertial coordinate directions.

The astrodynamic constant uncertainties $\Delta\mu_E$, $\Delta\mu_M$, and ΔR_{EM} were assumed normally distributed and independent, with zero means and standard deviations as given in table I.

Miss at the Moon with no guidance.- From a run of the computer program in which no observations or velocity corrections are made, the transition matrix from injection to perilune passage, $\Phi^*(t_m,t_o)$, is obtained. The elements of this matrix are the sensitivity coefficients which give the miss at the Moon for unit initial errors in the position and velocity components and the astrodynamic constants. Using this matrix and assuming that the rms values of the astrodynamic constant errors are the values given in table I, we can compute that the rms miss at the Moon due to these errors is about 202 km in position and 13.6 m/sec in velocity. By far the greatest portion of these errors is due to $\Delta\mu_E$.

For comparison, it might be noted that the miss due to the injection errors assumed (1 km and 1 m/sec in each direction) is 2660 km and 161 m/sec rms. Most of this miss is due to the injection velocity error.

Effect of the uncertainties on performance, with guidance.- The effect of the astrodynamic constant errors on system performance was measured by comparing computer runs made with and without the assumption of errors in the constants. Some of the results of these runs are shown in tables III and IV which give rms performance at perilune and at perigee, respectively.

The first five lines of table III show perilune performance with the 426-observation schedule of reference 2 for conditions of no errors, μ_E error only, μ_M error only, R_{EM} error only, and all three errors, so that the individual effects of the separate errors can be ascertained. The runs made with the other schedules were all either with or without all three errors.

The data in tables III and IV can be summarized as follows:

(a) At perilune the uncertainties, $\tilde{r}$ and $\tilde{v}$, are increased roughly 2-1/2 times by the astrodynamic constant errors when the reference 2 and long schedules are used. The percentage degradation for the short schedule with only 45 observations is a bit less primarily because the $\tilde{r}$ and $\tilde{v}$ are already larger on account of the fewer observations. However, the total degradation is greater for fewer observations because the astrodynamic constants are estimated more poorly. It is seen that $\tilde{r}$ is mostly affected by the R_{EM} error, and $\tilde{v}$ by the μ_M error.

(b) The miss quantities r_p and r at perilune are roughly doubled by the astrodynamic constant errors, but v is increased only 27 percent. The R_{EM} and μ_M errors produce most of the r_p variation, and the μ_M error produces most of the r and v increase.

(c) At perigee the degradation in performance is quite small, typically about 7 percent, except that perigee variation is almost entirely unaffected. Thus, what variation occurs is principally in time of arrival. The small loss in performance compared to performance at perilune is probably due to the fact that performance is already quite a bit poorer at perigee than at perilune even without the astrodynamic constant errors.

(d) The total applied ΔV is virtually unaffected on the outbound leg, what little increase there is being mostly due to $\Delta\mu_E$. On the return leg there is an increase of about 20 percent which is due primarily to the increased v at perilune.

Estimation of the astrodynamic constants.- An interesting by-product of the study described here is the improvement in knowledge of the astrodynamic constants as a result of incorporating them in the estimation process. The standard deviations of the errors in knowledge of μ_E, μ_M, and R_{EM} are obtained directly from the P matrix. These are tabulated in table V. It is seen that only μ_M, the most poorly known of the three constants percentagewise, is refined significantly. An improvement of about 5 or 6 to 1 is obtained even with relatively few observations (80 to 400) made using the short and long schedules during the circumlunar flight, whereas μ_E and R_{EM} can muster only 15 and 8 percent improvement, respectively.

Table V also shows that if only one constant is being estimated and the others are assumed known perfectly, then the indicated standard deviation of the error in estimate is smaller than when all three constants are assumed in error. This illustrates an important point; namely, that if in an error analysis some of the error sources are unaccounted for, the final estimate of the error statistics will always be optimistic (i.e., too small). Thus, it is well in cases where one is not sure all the errors have been adequately described to take a pessimistic view of the results obtained from the study.

Compensation for Mixed Bias and Random Observation Errors

The situation simulated for this study is the same as that employed in reference 1. The mission and injection conditions are the same as described in the previous section, but only the first few hours of flight are considered. The observation schedule consists of a sequence of observations of the Earth, starting one-half hour after injection, spaced 6 minutes apart for 2 hours. The situation is illustrated in figure 4. The uncorrelated observation errors assumed in the study have a standard deviation of 20 sec arc, independent of range, for each of the three measured angles. The bias error assumed has a standard deviation of 5 sec arc.

A digital computer program was written to implement the equations for estimating bias error developed in the Analysis section of this paper. Runs were then made simulating the situation described here, with bias on only one of the three angles at a time, for the purpose of identifying the effects of the location of the bias error.

Figure 5 shows the time history of the rms errors in estimating the vehicle's position and velocity, with and without an assumed bias on the half-subtended angle, γ. The difference between the two cases is seen to have increased to about 10 percent at the end of the observation period. Similar runs made with bias on the declination angle, α, and the right ascension angle, β, showed a nearly negligible effect (no more than 1 percent) due to bias, and these results are not shown.

In figure 6 are shown the time histories of the rms errors in estimating the biases for each of the three situations. It is seen that the estimates of α and β bias are improved scarcely at all by the 20 observations, whereas the estimate of γ bias is improved about 25 percent. The conclusion drawn from these results is that the observations assumed contain some information regarding γ bias but practically none regarding α and β bias.

Besides computing the statistical results given above, the computer program simulates an actual flight, with randomly selected injection errors and observation errors. The injection errors assumed for all the runs described here are:

$$\left.\begin{aligned} x_1 &= 0.495 \text{ km} \\ x_2 &= -0.886 \text{ km} \\ x_3 &= -1.001 \text{ km} \end{aligned}\right\} \text{position components}$$

$$\left.\begin{aligned} x_4 &= 0.281 \text{ m/sec} \\ x_5 &= 1.999 \text{ m/sec} \\ x_6 &= 0.194 \text{ m/sec} \end{aligned}\right\} \text{velocity components}$$

The random observation errors assumed are shown in figure 7. Time histories of the errors in estimating the position error r and the velocity error v are shown in figure 8 for three situations: (1) bias of 5 sec arc on γ but estimation of the bias not implemented, (2) bias of 5 sec arc on γ with estimation, and (3) no bias and no estimation thereof. It is seen that situation (1) gives the poorest results, as expected. Situation (3) of course gives the best results, and situation (2) generally lies between (1) and (3), being near (1) in the early part of the observation sequence, and tending to approach (3) at the end of the sequence as the estimate of γ bias is improved. Similar runs using α and β bias showed that the effect of these biases is negligible.

The corresponding time history of the γ bias estimate is shown in figure 9. Also shown are the time histories of α and β bias estimate for identical runs in which α or β bias was assumed instead of γ bias. The α and β bias estimates are seen to be essentially zero, reflecting again the fact that the assumed sequence of observations contains little information about these random variables. The γ estimate on the other hand shows substantial fluctuations. In the early part of the observation sequence the estimate is seen to be negative even though the actual bias is positive. This behavior is readily traced to the nature of the particular injection errors employed in this study. These errors cause the vehicle's distance from the Earth to be greater than nominal for all of the observation period. For instance, at the time of the fourth observation, 48 minutes after injection, the range is 8 km greater than the nominal 19,825, so that the half-subtended angle γ is 30 sec arc smaller than the nominal. So far as the instruments are concerned, it makes no difference whether this deviation of 30 sec arc is due to injection errors, bias, or random measurement error. The data-processing system of course apportions the difference among the various sources according to its knowledge of the statistics of the various errors. Since the actual deviation is negative, the system tends to estimate that all the error sources are producing negative error even though in this early part of the flight both the bias and random errors (see fig. 7) are actually positive. Later, as more data is accumulated, the system can more accurately apportion the errors, and the γ bias estimate becomes positive.

It might be noted that the bias error is never known very well (see fig. 6) during the simulated observation period, so the rather good estimate late in the observation period must be regarded as somewhat fortuitous. This behavior can be explained by the character of the particular sample of random γ error used in the run, which happens to be substantially biased (see fig. 7); the mean value of the random error for the first 14 observations is 8.5 sec arc. This is not to say that the particular sample employed is highly extraordinary - it is no more unlikely for instance than a run of heads in tossing a coin. Nevertheless, the biased sample does have a typical effect on the estimation process. In the first place, it tends to be interpreted by the system as a bias error and results in a bias estimate larger than would normally be expected, as already noted. In the second place, it can also appear to the system to be the result of injection errors since these produce a bias-type deviation from nominal. The errors in estimating vehicle position and velocity thus tend to be larger than normal when such a sequence occurs. This effect shows quite clearly in figure 8, where it is seen that for the first eight observations the estimates with bias estimation are actually poorer than the estimates obtained with bias estimation omitted.

Bias Error Off-Design System Performance

For the study of off-design performance the situation simulated was that of the complete circumlunar mission described earlier. Three-angle observations (α, β, and γ) were assumed as before, corrupted by random errors (i.e., uncorrelated from one observation to the next) and also bias errors. The trajectory estimation system employed was optimal for the uncorrelated errors only - that is, it was assumed that the presence of the bias error was unrecognized.

To obtain numerical results, the equations for off-design performance given in the Analysis section were implemented in a digital computer program. Runs were made with the short and long schedules shown in figures 2 and 3, with uncorrelated observation error having a range-dependent standard deviation as given by equation (80).

Bias error was assumed to have an rms value of first 5 sec arc and then in later runs 10 sec arc. This error was applied in some cases to all three angles and in other cases separately to each of the angles α, β, and γ to ascertain the effect of location of the error. To avoid excessive complexity, it was assumed that the same bias exists on a particular angle measurement (e.g., α) whether this angle is associated with a Moon or an Earth observation. This is probably not a particularly realistic model for bias error, but it should suffice to give some general results regarding the effects of bias.

The gross effect of bias on the estimation system is shown in figure 10, which shows the time histories of the rms errors in predicting the end-point miss for situations of no bias, 5 sec arc bias, and 10 sec arc bias on all three angles. The short observation schedule (80 observations) was used for these runs. For the first part of the flight, the end point is defined as the time of nominal perilune - that is, the figure shows the rms error in predicting the vehicle's position at the time of nominal perilune. For the last part of the flight the end-point is the time of nominal virtual perigee. The data shown is correct only at the times of the observations; the lines connecting these data points are used only for clarity.

It is seen that the bias has a substantial effect on system performance, an effect which increases rapidly as the bias increases. The effect is distinctly different in different portions of the flight. It is of particular interest to note that with bias present the rms prediction error actually increases when some observations are made, whereas if an optimal data processing system were employed this quantity would always decrease. The physical interpretation of this phenomenon is that too much weight is being given to the observations because the system believes the observations to be more accurate than they really are. In some instances the error introduced in this manner is, statistically speaking, greater than the reduction in error obtained by utilizing the information in the observation, and the rms prediction error experiences a net increase. Apparently, it would be better not to use such observations at all, but under the assumed circumstances the system has no knowledge that the bias errors exist and therefore has no basis upon which to make a decision as to the appropriateness of the particular observations.

Shown in figures 11, 12, and 13 are the effects of bias on only one angle at a time, the bias level being 10 sec arc rms in each case. It is seen that β bias has by far the most pronounced effect, accounting for virtually all the performance degradation observed in the case of bias on all three angles. The γ bias is seen to be quite unimportant and the α bias of some importance. This result is consistent with the results of other unreported studies which have shown that angle measurements in the plane of the vehicle trajectory yield the most navigation information. Since the plane of the trajectory used in this study is close to the reference equatorial plane, β is essentially an in-plane angle. Thus, β measurements receive the most weight in the processing of data, so that unrecognized errors in β will produce the greatest performance degradation.

For α measurements, bias appears to have the most detrimental effect in the middle regions of cislunar space. For β measurements, the effect of bias is greatest when β measurements are being made of the more distant of the two bodies (Earth and Moon). For γ measurements the greatest effect of bias occurs close to Earth or Moon when the near body is being observed. This is consistent with the results of the previous section where only observations of the Earth in the early part of the flight were considered, and γ bias was seen to have a predominant effect in this region.

The preceding results have shown the degradation in system performance which occurs either when an unrecognized bias error exists or when the existence of a bias error is known but ignored in designing the system. If the situation is the former then the results given are simply an off-design performance analysis. However, if the situation is the latter, the question arises as to whether the system design can be altered, without increasing its complexity, to reduce the performance degradation. A simple remedy that may be considered is to assume that the random error is larger than it actually is, which means merely using a larger Q_1 matrix in the computations. The logic behind such a design compromise is that if a larger Q_1 matrix is used the P matrix computed in the estimation process is not reduced so rapidly as observations are made. The system then pays more attention to later observations than it would otherwise so that the false information in early observations is not so damaging to the performance.

To illustrate this principle, a computer run was made in which there was no bias error (i.e., $Q_2 = 0$) but Q_1 was increased so that the basic random error assumed was 20 sec arc. That is, the standard deviation of random observation error is given by the formula

$$\sigma_n = \sqrt{(20)^2 + (0.001\ \gamma)^2}\ \text{sec arc} \tag{94}$$

rather than the expression given earlier (eq. (93)). The rms prediction error for this run is plotted in figure 14 together with the uncompensated 5 arc sec bias run repeated from figure 10. It is seen that the performances for these two situations match fairly well. That is, the performance of a system with no bias and properly compensated basic random error of 20 sec arc (eq. (94)) is similar to the performance of a system with uncompensated 5 sec arc bias and properly compensated basic random error of 10 sec arc (eq. (93)).

The next step would be to try using Q_1 as given by equation (94) in the bias error situation, which should improve the performance some. No results have been obtained to show the amount of improvement, but it is expected that the performance would be about the same as that of an optimal system with a basic random error of 15 sec arc.

When the long schedule (a total of 400 observations) is used instead of the short schedule, a more severe degradation in performance is produced by uncompensated bias. This effect is shown in figure 15, which shows the performance of the estimation system using the long schedule for no bias error and for 10 sec arc bias. It is seen that the degradation due to bias is so severe, that the long schedule with bias actually has poorer performance than the short schedule with bias (except in the region near perilune). The reason for this is that the P' matrix becomes small much more rapidly in this case than with the short schedule, and the erroneous information obtained from the biased early observations has less chance of being corrected by the subsequent observations. As in the case of the short schedule, increasing the size of the Q_1 employed in the computation should improve the performance, perhaps substantially since the bias-produced degradation is so large.

The ultimate measures of the performance of the guidance system are the end-point miss and the total velocity correction employed (which is equivalent to fuel required). For a guidance system employing the estimation scheme which has been described these quantities are given in tables VI and VII. Results are shown for each of the bias conditions considered. The guidance law used for computing the velocity corrections is the same as described earlier in reference 2 (using fixed-time-of-arrival principles), each correction being computed from the estimated state vector at the time of the correction. Errors in implementing the velocity corrections were assumed to be 0.1 m/sec in magnitude and 0.5° in direction, rms. Five corrections were made in each run, three on the outbound flight and two on the return. The time schedule of corrections, shown in figures 2 and 3, is slightly different for the long and short observation schedules.

The rms miss data in tables VI and VII is given in terms of the total position and velocity deviations from the nominal trajectory at the perilune and perigee end points. The rms variations in actual perilune and perigee distances are also given, the latter being significant for indicating the probability of safe atmospheric entry in terms of the corridor concept. It is seen that the perigee deviations for all situations are sufficiently small to virtually insure safe entry. Since perigee and perilune deviations are quite small compared to the total position deviations, the indication is that most of the miss occurs along the flight path - that is, the miss is mostly a deviation in the time of arrival at the end point. The velocity deviation at perilune is mostly a deviation known to the guidance system since the velocity uncertainty (i.e., the rms error in the knowledge of the velocity) is seen always to be an order of magnitude smaller than the deviation. This occurs because the guidance law employed corrects only the end-point position deviation, leaving a known but uncorrected velocity deviation. The perilune velocity deviation must, of course, be corrected on the return leg of the flight since it produces an end-point position deviation at the Earth. This accounts for a large part of the ΔV required on the return leg. The velocity deviation at the Earth end-point can be regarded as a deviation in reentry velocity which is an initial condition for the terminal guidance system.

The differences in miss and ΔV performance for the different bias assumptions are seen to be very nearly the same as the difference already illustrated in the time history plots of prediction uncertainty, figures 10-14. For the short schedule the 10 sec arc bias produces a 40- to 50-percent increase in perilune position deviation and ΔV. The velocity deviation at the Moon is doubled. At the Earth end-point the miss has been nearly tripled but the perigee altitude deviation is increased only about 25 percent. The return ΔV is up 85 percent, principally because of the substantial increase in perilune velocity deviation.

For the long schedule, position and velocity deviations at perilune are tripled by the 10 sec arc bias but perilune altitude deviation is up only slightly, and ΔV is increased 65 percent. For the return flight, end-point position and velocity deviations are increased roughly six times and ΔV is almost tripled, but perigee altitude variation is merely doubled, so that safe reentry could still be effected. Note that perigee end-point conditions and total ΔV for the long schedule are worse than for the short schedule with bias indicating that there would be no point in using the long schedule in the presence of such a bias unless the system is modified in some way to take the bias into account.

CONCLUDING REMARKS

Equations have been developed which show (1) how to include astrodynamic constant uncertainties and bias-type errors in the estimation process, and (2) how to compute the performance of a system subjected to unrecognized or ignored bias errors. The demonstrated techniques can be extended in a more or less obvious manner to handle any type of error model desired in the estimation process, and to treat many other types of off-design analysis problems. In principle, extending the estimation process to more complicated error models requires augmenting the state vector with additional error components. For the analysis of off-design performance it is seen that one must in general arrange to compute the covariance matrix $E\hat{x}\tilde{x}^T$, which is zero for optimal estimation but not so for off-design situations.

The numerical results given in the report show that the astrodynamic constant uncertainties affect system performance in only a minor way. Nevertheless, this type of secondary error source does distinctly limit system performance at the lower end and must be given thoughtful consideration when quoting performance capabilities.

In regard to the effects of bias errors, an exhaustive analysis has not been attempted. However, the numerical results indicate that bias can be expected to have approximately the same effect on system performance as uncorrelated errors of the same magnitude. Also, the indication is that use of a suitable uncorrelated error model as an approximation to the true model will yield adequate though not optimum results.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Oct. 2, 1963

APPENDIX

CONTRACTION OF THE ESTIMATION EQUATIONS FOR UNCORRELATED OBSERVATION ERRORS

Estimation of the Astrodynamic Constants

The estimated state vector is defined as

$$\hat{x}^* = \begin{Bmatrix} \hat{x} \\ \hat{e} \\ \hat{w} \end{Bmatrix} \tag{A1}$$

and the extrapolation of the estimate from the time of one observation to the next is of the form

$$\begin{Bmatrix} \hat{x} \\ \hat{e} \\ \hat{w} \end{Bmatrix}_{t_k} = \begin{bmatrix} \Phi_{xx} & \Phi_{xe} & 0 \\ 0 & I & 0 \\ 0 & 0 & \Phi_w \end{bmatrix} \begin{Bmatrix} \hat{x} \\ \hat{e} \\ \hat{w} \end{Bmatrix}_{t_{k-1}} \tag{A2}$$

The assumption that the errors in successive observations are uncorrelated is equivalent to assuming $\Phi_w = 0$ in equation (A2); that is,

$$\hat{w}_{k-1}(t_k) = \Phi_w(t_k,t_{k-1})\hat{w}_{k-1}(t_{k-1}) = 0 \tag{A3}$$

Thus, equation (A2) can be contracted to the form

$$\begin{Bmatrix} \hat{x} \\ \hat{e} \end{Bmatrix}_{t_k} = \begin{bmatrix} \Phi_{xx} & \Phi_{xe} \\ 0 & I \end{bmatrix} \begin{Bmatrix} \hat{x} \\ \hat{e} \end{Bmatrix}_{t_{k-1}} \tag{A4}$$

For processing the kth observation, the equation is

$$\begin{Bmatrix} \hat{x} \\ \hat{e} \\ \hat{w} \end{Bmatrix}_k = \begin{Bmatrix} \hat{x} \\ \hat{e} \\ \hat{w} \end{Bmatrix}_{k-1} + \begin{bmatrix} K_x \\ K_e \\ K_w \end{bmatrix} \left\{ y - \begin{bmatrix} H & 0 & \Gamma \end{bmatrix} \begin{Bmatrix} \hat{x} \\ \hat{e} \\ \hat{w} \end{Bmatrix}_{k-1} \right\} \tag{A5}$$

Since $\hat{w}$ for use in (A5), as determined by (A3), is zero, equation (A5) is

$$\begin{Bmatrix} \hat{x} \\ \hat{e} \\ \hat{w} \end{Bmatrix}_k = \begin{Bmatrix} \hat{x} \\ \hat{e} \\ 0 \end{Bmatrix}_{k-1} + \begin{bmatrix} K_x \\ K_e \\ K_w \end{bmatrix} (y - H\hat{x}_{k-1}) \tag{A6}$$

from which it is seen that

$$\hat{w}_k = K_w(y - H\hat{x}_{k-1}) \tag{A7}$$

However, since the next application of equation (A3) will result, by assumption, in $\hat{w}_{k+1}(t_k) = 0$, there is no purpose to implementing equation (A7), and (A5) may be contracted to

$$\begin{Bmatrix} \hat{x} \\ \hat{e} \end{Bmatrix}_k = \begin{Bmatrix} \hat{x} \\ \hat{e} \end{Bmatrix}_{k-1} + \begin{bmatrix} K_x \\ K_e \end{bmatrix} (y - H\hat{x}_{k-1}) \tag{A8}$$

A similar method of analysis can be applied to the computation of the weighting matrix, K. For updating the P matrix between observations, we have

$$\begin{bmatrix} P_x & P_{xe} & P_{xw} \\ P_{ex} & P_e & P_{ew} \\ P_{wx} & P_{we} & P_w \end{bmatrix}_{t_k} = \begin{bmatrix} \Phi_{xx} & \Phi_{xe} & 0 \\ 0 & I & 0 \\ 0 & 0 & \Phi_w \end{bmatrix} \begin{bmatrix} P_x & P_{xe} & P_{xw} \\ P_{ex} & P_e & P_{ew} \\ P_{wx} & P_{we} & P_w \end{bmatrix}_{t_{k-1}}$$

$$\begin{bmatrix} \Phi_{xx}^T & 0 & 0 \\ \Phi_{xe}^T & I & 0 \\ 0 & 0 & {\Phi_w}^T \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & U' \end{bmatrix} \tag{A9}$$

If $\Phi_w = 0$ for uncorrelated observation errors, then it is readily seen that

$$\left.\begin{aligned}
P_x(t_k) &= \Phi_{xx}P_x(t_{k-1})\Phi_{xx}^T + \Phi_{xe}P_e(t_{k-1})\Phi_{xe}^T + \Phi_{xe}P_{ex}(t_{k-1})\Phi_{xx}^T \\
&\quad + \Phi_{xx}P_{xe}(t_{k-1})\Phi_{xe}^T \\
P_{xe}(t_k) &= \Phi_{xx}P_{xe}(t_{k-1}) + \Phi_{xe}P_e(t_{k-1}) \\
P_e(t_k) &= P_e(t_{k-1}) \\
P_w(t_k) &= U' \\
P_{xw}(t_k) &= P_{ew}(t_k) = 0
\end{aligned}\right\} \tag{A10}$$

For the computation of $K(t_k)$, we then have

$$MP^*M^T = [H \quad 0 \quad \Gamma] \begin{bmatrix} P_x & P_{xe} & 0 \\ P_{ex} & P_e & 0 \\ 0 & 0 & U' \end{bmatrix}_{t_k} \begin{bmatrix} H^T \\ 0 \\ \Gamma^T \end{bmatrix}$$

$$= [HP_xH^T + \Gamma U'\Gamma^T] \tag{A11}$$

Then

$$\begin{bmatrix} K_x \\ K_e \\ K_w \end{bmatrix} = \begin{bmatrix} P_x & P_{xe} & 0 \\ P_{ex} & P_e & 0 \\ 0 & 0 & U' \end{bmatrix}_{t_k} \begin{bmatrix} H^T \\ 0 \\ \Gamma^T \end{bmatrix} [HP_xH^T + \Gamma U'\Gamma^T]^{-1}$$

$$= \begin{bmatrix} P_xH^T \\ P_{ex}H^T \\ U'\Gamma^T \end{bmatrix} [HP_xH^T + Q]^{-1} \tag{A12}$$

where

$$Q = \Gamma U'\Gamma^T$$

Finally, the change in P when the observation is processed is

$$\Delta P = P_{k-1} - P_k = KMP_{k-1}$$

$$= \begin{bmatrix} K_x \\ K_e \\ \hline K_w \end{bmatrix} [H \quad 0 \quad \Gamma] \left[\begin{array}{cc:c} P_x & P_{xe} & 0 \\ P_{ex} & P_e & 0 \\ \hdashline 0 & 0 & U' \end{array}\right]_{t_k} \tag{A13}$$

Because of the null submatrices in $P(t_k)$, the portion of ΔP related to the observation errors (to the right and below the dotted lines) is uncoupled from the rest of ΔP. Since this portion of ΔP is not needed, and since K_w is not needed, equations (A9), (A12), and (A13) may be contracted as follows:

$$\begin{bmatrix} P_x & P_{xe} \\ P_{ex} & P_e \end{bmatrix}_{t_k} = \begin{bmatrix} \Phi_{xx} & \Phi_{xe} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_x & P_{xe} \\ P_{ex} & P_e \end{bmatrix}_{t_{k-1}} \begin{bmatrix} \Phi_{xx}^T & 0 \\ \Phi_{xe}^T & I \end{bmatrix} \tag{A14}$$

$$\begin{bmatrix} K_x \\ K_e \end{bmatrix} = \begin{bmatrix} P_x \\ P_{ex} \end{bmatrix}_{t_k} H^T[HP_xH^T + Q]^{-1} \tag{A15}$$

$$\begin{bmatrix} \Delta P_x & \Delta P_{xe} \\ \Delta P_{ex} & \Delta P_e \end{bmatrix} = \begin{bmatrix} K_x \\ K_e \end{bmatrix} [H \quad 0] \begin{bmatrix} P_x & P_{xe} \\ P_{ex} & P_e \end{bmatrix}_{t_k} \tag{A16}$$

Estimation of Mixed Random and Bias Observation Error

The estimated state vector is defined as

$$\hat{x}^* = \begin{Bmatrix} \hat{x} \\ \hat{n}_1 \\ \hat{n}_2 \end{Bmatrix} \tag{A17}$$

and the extrapolation from the time of one observation to the next is of the form

$$\begin{Bmatrix} \hat{x} \\ \hat{n}_1 \\ \hat{n}_2 \end{Bmatrix}_{t_k} = \begin{bmatrix} \Phi_x & 0 & 0 \\ 0 & \Phi_1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \hat{x} \\ \hat{n}_1 \\ \hat{n}_2 \end{Bmatrix}_{t_{k-1}} \tag{A18}$$

Under the assumption that the n_1 error in the kth observation is uncorrelated with that in the (k-1)st observation (i.e., $\Phi_1 = 0$), equation (A18) can be contracted to

$$\begin{Bmatrix} \hat{x} \\ \hat{n}_2 \end{Bmatrix}_{t_k} = \begin{bmatrix} \Phi_x & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} \hat{x} \\ \hat{n}_2 \end{Bmatrix}_{t_{k-1}} \tag{A19}$$

For processing the kth observation, the contracted equation is

$$\begin{Bmatrix} \hat{x} \\ \hat{n}_2 \end{Bmatrix}_{k} = \begin{Bmatrix} \hat{x} \\ \hat{n}_2 \end{Bmatrix}_{k-1} + \begin{bmatrix} K_x \\ K_{n_2} \end{bmatrix} \left\{ y - [H \; \vdots \; I] \begin{Bmatrix} \hat{x} \\ \hat{n}_2 \end{Bmatrix}_{k-1} \right\} \tag{A20}$$

For updating the P matrix between observations we have

$$\begin{bmatrix} P_x & P_{xn_1} & P_{xn_2} \\ P_{n_1x} & P_{n_1} & P_{n_1n_2} \\ P_{n_2x} & P_{n_2n_1} & P_{n_2} \end{bmatrix}_{t_k} = \begin{bmatrix} \Phi_x & 0 & 0 \\ 0 & \Phi_1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_x & P_{xn_1} & P_{xn_2} \\ P_{n_1x} & P_{n_1} & P_{n_1n_2} \\ P_{n_2x} & P_{n_2n_1} & P_{n_2} \end{bmatrix}_{t_{k-1}}$$

$$\begin{bmatrix} \Phi_1^T & 0 & 0 \\ 0 & \Phi_1^T & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & U' & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A21}$$

where U' is the covariance matrix of $u'(t_k,t_{k-1})$. If $\Phi_n = 0$, then it is seen that

$$P(t_k) = \begin{bmatrix} \Phi_x P_x \Phi_x^T & 0 & \Phi_x P_{xn_2} \\ 0 & Q_1 & 0 \\ P_{n_2x}\Phi_x^T & 0 & P_{n_2} \end{bmatrix}_{t_{k-1}} \tag{A22}$$

Here, U' has been replaced by Q_1 which, because of the assumption of uncorrelated observation error, is dependent only upon the time of the present observation and can be a stored quantity.

For the computation of $K(t_k)$, we then have

$$MPM^T = [H \quad I \quad I] \begin{bmatrix} P_x & 0 & P_{xn_2} \\ 0 & Q_1 & 0 \\ P_{n_2x} & 0 & P_{n_2} \end{bmatrix}_{t_k} \begin{bmatrix} H^T \\ I \\ I \end{bmatrix}$$

$$= (HP_xH^T + P_{n_2x}H^T + HP_{xn_2} + P_{n_2} + Q_1) \tag{A23}$$

Then

$$\begin{bmatrix} K_x \\ K_{n_1} \\ K_{n_2} \end{bmatrix} = \begin{bmatrix} P_x & 0 & P_{xn_2} \\ 0 & Q_1 & 0 \\ P_{n_2x} & 0 & P_{n_2} \end{bmatrix} \begin{bmatrix} H^T \\ I \\ I \end{bmatrix} [A]$$

$$= \begin{bmatrix} (P_xH^T + P_{xn_2}) \\ (Q_1) \\ (P_{n_2x}H^T + P_{n_2}) \end{bmatrix} [A] \tag{A24}$$

where $A = (MPM^T)^{-1}$ from equation (A23). Finally, the change in P when the observation is processed is

$$\Delta P = P_{k-1} - P_k = KMP$$

$$= \begin{bmatrix} K_x \\ K_{n_1} \\ K_{n_2} \end{bmatrix} [H \quad I \quad I] \begin{bmatrix} P_x & 0 & P_{xn_2} \\ 0 & Q_1 & 0 \\ P_{n_2x} & 0 & P_{n_2} \end{bmatrix}_{t_k} \tag{A25}$$

Since K_{n_1} is not needed, it is seen that equations (A21), (A24), and (A25) can be contracted as follows:

$$\begin{bmatrix} P_x & P_{xn_2} \\ P_{n_2x} & P_{n_2} \end{bmatrix}_{t_k} = \begin{bmatrix} \Phi_x & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_x & P_{xn_2} \\ P_{n_2x} & P_{n_2} \end{bmatrix}_{t_{k-1}} \begin{bmatrix} \Phi_x{}^T & 0 \\ 0 & I \end{bmatrix} \tag{A26}$$

$$\begin{bmatrix} K_x \\ K_{n_2} \end{bmatrix} = \begin{bmatrix} P_xH^T + P_{xn_2} \\ P_{n_2x} + P_{n_2} \end{bmatrix} [A] \tag{A27}$$

$$\begin{bmatrix} \Delta P_x & \Delta P_{xn_2} \\ \Delta P_{n_2x} & \Delta P_{n_2} \end{bmatrix} = \begin{bmatrix} K_x \\ K_{n_2} \end{bmatrix} [H \quad I] \begin{bmatrix} P_x & P_{xn_2} \\ P_{n_2x} & P_{n_2} \end{bmatrix}_{t_k} \tag{A28}$$

REFERENCES

1. Smith, Gerald L., Schmidt, Stanley F., and McGee, Leonard A.: Application of Statistical Filter Theory to the Optimal Estimation of Position and Velocity On Board a Circumlunar Vehicle. NASA TR R-135, 1962.

2. McLean, John D., Schmidt, Stanley F., and McGee, Leonard A.: Optimal Filtering and Linear Prediction Applied to a Midcourse Navigation System for the Circumlunar Mission. NASA TN D-1208, 1962.

3. Makemson, Maud W., Baker, Robert M. L., Jr., and Westrom, George W.: Analysis and Standardization of Astrodynamic Constants. AAS Preprint 61-27, 1961.

4. Kalman, R. E.: A New Approach to Linear Filtering and Prediction Problems. Trans. ASME Journal of Basic Engineering, vol. 82, March 1960, pp. 35-45.

5. Kalman, R. E.: New Methods and Results in Linear Prediction and Filtering Theory. RIAS Tech. Rep. 61-1, 1960.

TABLE I.- ERRORS ASSUMED

Error	Magnitude	Percentage	Max $\Delta\ddot{X}(g)$
$\Delta\mu_E$	$\pm 9\times 10^9$ m^3/sec^2	0.002	2×10^{-5}
$\Delta\mu_M$	$\pm 3\times 10^9$ m^3/sec^2	.06	1×10^{-4}
$\Delta\mu_S$	$\pm 4\times 10^{16}$ m^3/sec^2	.03	9×10^{-10}
ΔR_{EM}	± 2 km	.0005	3.7×10^{-4}
ΔR_{ES}	$\pm 4\times 10^4$ km	.03	1.9×10^{-9}
ΔJ	$\pm 5\times 10^{-8}$	.003	4.5×10^{-8}

TABLE II.- H-MATRIX PARTIALS

	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial Y}$	$\frac{\partial}{\partial Z}$	$\frac{\partial}{\partial R_{EM}}$
α_E	$\frac{X_{EV}Z_{EV}}{R_{EV}^2(X_{EV}^2 + Y_{EV}^2)^{1/2}}$	$\frac{Y_{EV}Z_{EV}}{R_{EV}^2(X_{EV}^2 + Y_{EV}^2)^{1/2}}$	$\frac{Z_{EV}^2 - R_{EV}^2}{R_{EV}^2(X_{EV}^2 + Y_{EV}^2)^{1/2}}$	0
β_E	$\frac{-Y_{EV}}{X_{EV}^2 + Y_{EV}^2}$	$\frac{X_{EV}}{X_{EV}^2 + Y_{EV}^2}$	0	0
γ_E	$\frac{-R_E X_{EV}}{R_{EV}^2(R_{EV}^2 - R_E^2)^{1/2}}$	$\frac{-R_E Y_{EV}}{R_{EV}^2(R_{EV}^2 - R_E^2)^{1/2}}$	$\frac{-R_E Z_{EV}}{R_{EV}^2(R_{EV}^2 - R_E^2)^{1/2}}$	0
α_M	$\frac{X_{MV}Z_{MV}}{R_{MV}^2(X_{MV}^2 + Y_{MV}^2)^{1/2}}$	$\frac{Y_{MV}Z_{MV}}{R_{MV}^2(X_{MV}^2 + Y_{MV}^2)^{1/2}}$	$\frac{Z_{MV}^2 - R_{MV}^2}{R_{MV}^2(X_{MV}^2 + Y_{MV}^2)^{1/2}}$	$\frac{Z_{MV}/R_{MV}^2(\bar{R}_{MV}\cdot\bar{R}_{EM}) + Z_{EM}}{R_{EM}(X_{MV}^2 + Y_{MV}^2)^{1/2}}$
β_M	$\frac{-Y_{MV}}{X_{MV}^2 + Y_{MV}^2}$	$\frac{X_{MV}}{X_{MV}^2 + Y_{MV}^2}$	0	$\frac{Y_{EV}X_{EM} - X_{EV}Y_{EM}}{R_{EM}(X_{MV}^2 + Y_{MV}^2)}$
γ_M	$\frac{-R_M X_{MV}}{R_{MV}^2(R_{MV}^2 - R_M^2)^{1/2}}$	$\frac{-R_M Y_{MV}}{R_{MV}^2(R_{MV}^2 - R_M^2)^{1/2}}$	$\frac{-R_M Z_{MV}}{R_{MV}^2(R_{MV}^2 - R_M^2)^{1/2}}$	$\frac{R_M(\bar{R}_{EV}\cdot\bar{R}_{MV})}{R_{EM}R_{MV}^2(R_{MV}^2 - R_M^2)^{1/2}}$

R_E radius of Earth

R_M radius of Moon

$\bar{R}_{EM}$ Earth-Moon distance $= \begin{Bmatrix} X_{EM} \\ Y_{EM} \\ Z_{EM} \end{Bmatrix}$

$\bar{R}_{EV}$ Earth-vehicle vector $= \begin{Bmatrix} X_{EV} \\ Y_{EV} \\ Z_{EV} \end{Bmatrix}$

$\bar{R}_{MV}$ Moon-vehicle vector $= \bar{R}_{EV} - \bar{R}_{EM} = \begin{Bmatrix} X_{MV} \\ Y_{MV} \\ Z_{MV} \end{Bmatrix}$

TABLE III.- PERILUNE PERFORMANCE - RMS VALUES

Schedule (No. of observations)	Errors	Miss			Uncertainty		Total applied ΔV, m/sec
		Perilune variation, r_p, km	r, km	v, m/sec	$\tilde{r}$, km	$\tilde{v}$, m/sec	
Ref. 2 (426)	None	1.67	4.76	1.17	0.79	0.077	9.76
Ref. 2 (426)	μ_E	1.71	4.79	1.18	.82	.082	9.81
Ref. 2 (426)	μ_M	2.40	8.96	1.46	.91	.185	9.78
Ref. 2 (426)	R_{EM}	2.72	5.23	1.19	1.98	.083	9.78
Ref. 2 (426)	μ_E, μ_M, R_{EM}	3.29	9.29	1.49	2.11	.187	9.85
Long (225)	None	1.66	5.80	1.23	.77	.075	9.37
Long (225)	μ_E, μ_M, R_{EM}	3.32	9.20	1.52	2.21	.207	9.50
Short (45)	None	2.39	10.62	1.73	1.79	.170	10.86
Short (45)	μ_E, μ_M, R_{EM}	3.92	13.06	1.96	2.96	.354	10.96

TABLE IV.- PERIGEE PERFORMANCE - RMS VALUES

Schedule (No. of observations)	Errors	Miss: Perilune variation, r_p, km	Miss: r, km	Miss: v, m/sec	Uncertainty: $\tilde{r}$, km	Uncertainty: $\tilde{v}$, m/sec	Total applied ΔV, m/sec
Short (80)	None	1.34	26.1	24.2	20.3	17.7	3.8
Short (80)	With	1.35	28.3	26.0	21.0	18.3	4.6
Long (400)	None	1.13	16.0	16.1	12.3	10.7	3.0
Long (400)	With	1.14	17.3	18.1	12.7	11.1	3.5

TABLE V.- RMS ERRORS IN THE ESTIMATION OF THE ASTRODYNAMIC CONSTANTS

Condition	Time	σ_{μ_E}, m³/sec²	σ_{μ_M}, m³/sec²	$\sigma_{R_{EM}}$, km	Schedule (No. of observations)
	At injection	9×10^9	3×10^9	2	(0)
Only $\Delta\mu_E$	At perilune	6.31×10^9			Ref. 2 (426)
Only $\Delta\mu_M$			1.30×10^9		
Only ΔR_{EM}				1.62	
All three errors		7.13×10^9	1.32×10^9	1.78	
		7.95×10^9	1.57×10^9	1.87	Long (225)
		8.58×10^9	2.31×10^9	1.95	Short (45)
	At perigee	8.45×10^9	$.646\times10^9$	1.93	Short (80)
		7.61×10^9	$.457\times10^9$	1.84	Long (400)

TABLE VI.- OFF-DESIGN RESULTS AT PERILUNE - RMS VALUES

Schedule (No. of observations)	Bias condition	Miss			Uncertainty		Total ΔV, m/sec
		Perilune variation, r_p, km	r, km	v, m/sec	$\tilde{r}$, km	$\tilde{v}$, m/sec	
80	No bias	2.4	10.6	1.7	1.8	0.17	10.9
80	5 sec α,β,γ	2.8	12.2	2.3	1.9	.19	12.5
80	10 sec α,β,γ	3.8	16.0	3.5	2.3	.24	15.4
80	10 sec on α	2.6	11.1	1.9	1.9	.20	11.2
80	10 sec on β	3.7	15.5	3.4	2.2	.22	15.2
80	10 sec on γ	2.4	10.8	1.7	1.8	.18	10.9
400	No bias	1.7	5.8	1.2	.77	.08	9.4
400	10 sec α,β,γ	1.9	17.1	4.8	1.4	.16	15.6

TABLE VII.- OFF-DESIGN RESULTS AT PERIGEE - RMS VALUES

Schedule (No. of observations)	Bias condition	Miss			Uncertainty		Total ΔV, m/sec
		Perigee variation, r_p, km	r, km	v, m/sec	$\tilde{r}$, km	$\tilde{v}$, m/sec	
80	No bias	1.3	26	24	20	18	3.8
80	5 sec α,β,γ	1.4	44	39	26	23	4.9
80	10 sec α,β,γ	1.6	76	66	39	34	7.1
80	10 sec on α	1.5	36	32	23	20	4.1
80	10 sec on β	1.5	71	62	38	33	7.0
80	10 sec on γ	1.4	26	24	21	18	3.8
400	No bias	1.1	16	16	12	11	3.0
400	10 sec α,β,γ	2.3	103	89	54	47	8.5

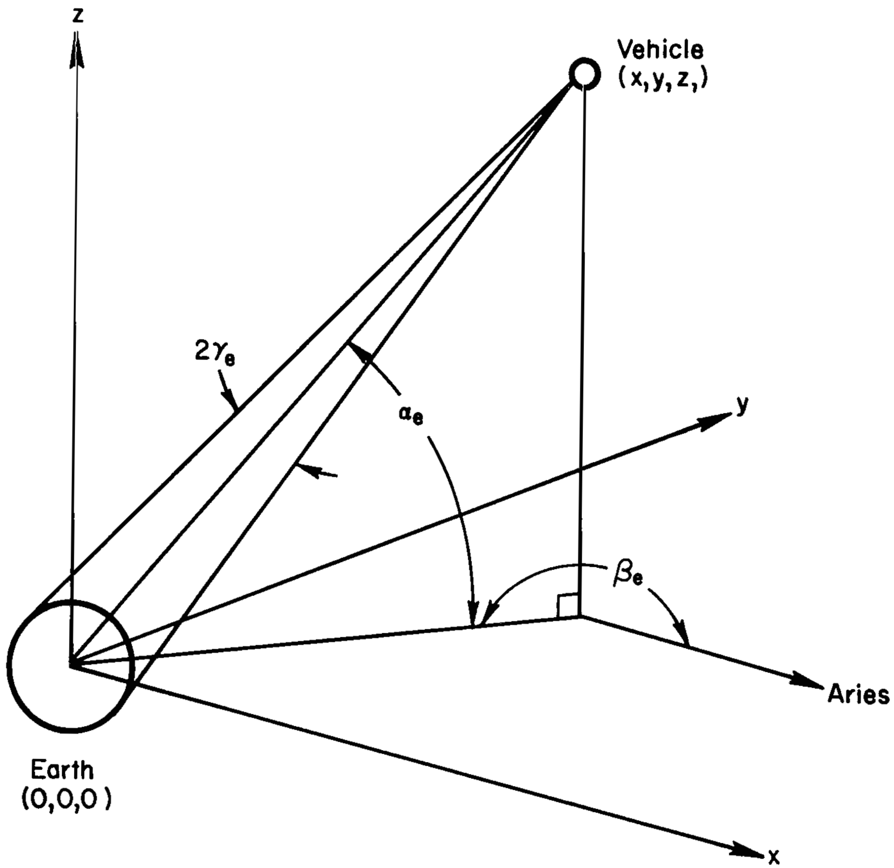

Figure 1.- Angles measured in an Earth observation.

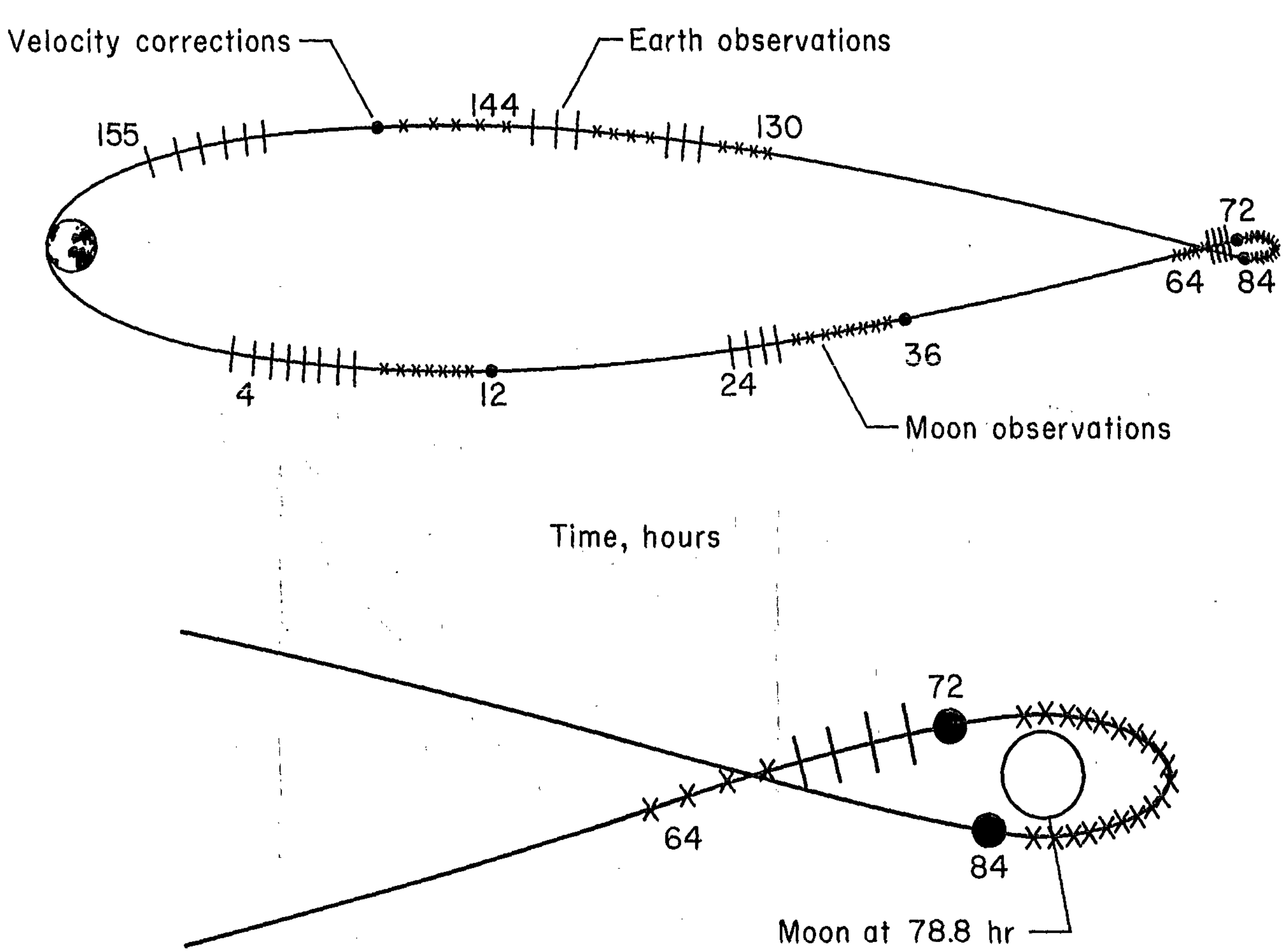

Figure 2.- Short observation and velocity correction schedule.

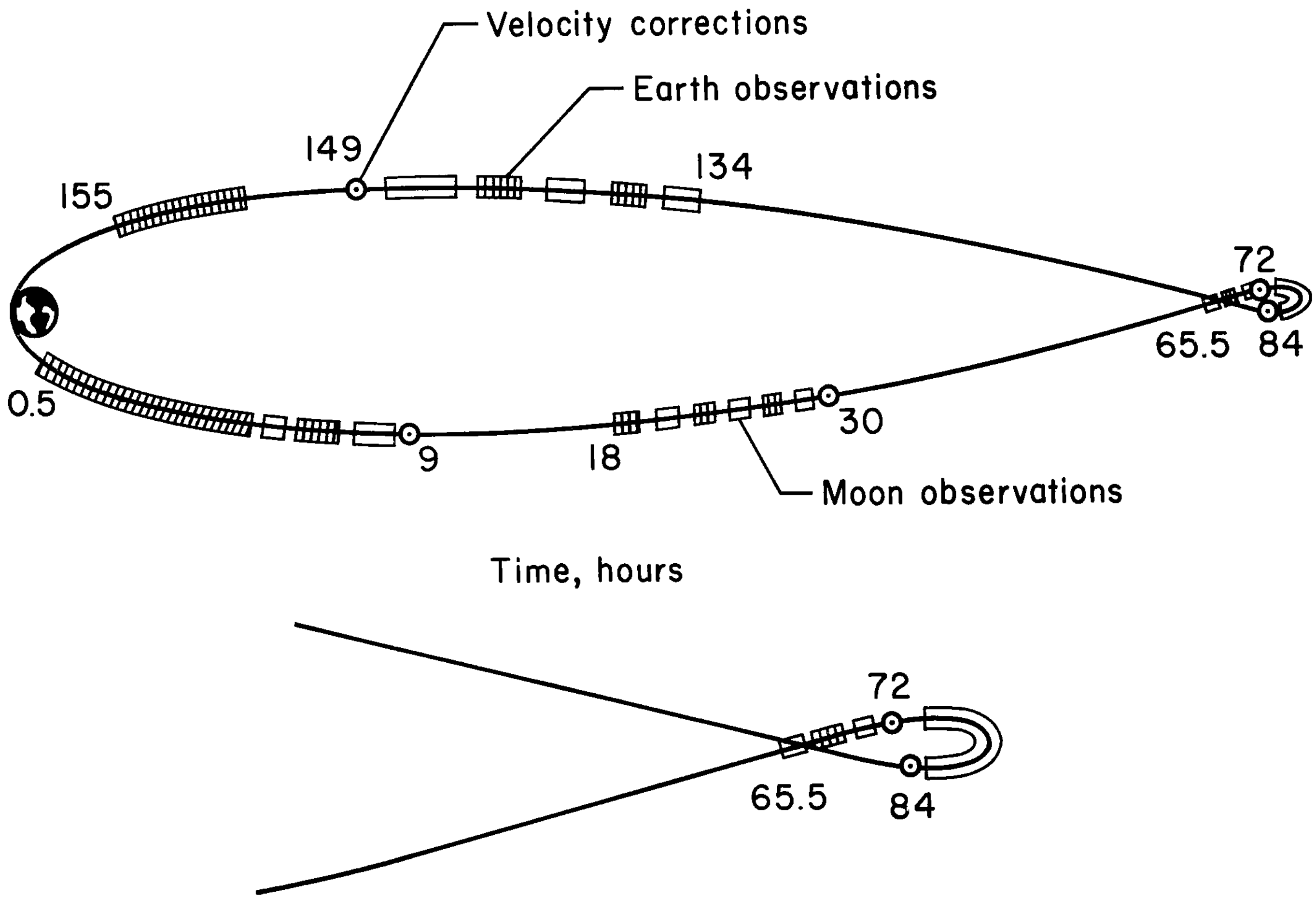

Figure 3.- Long observation and velocity correction schedule.

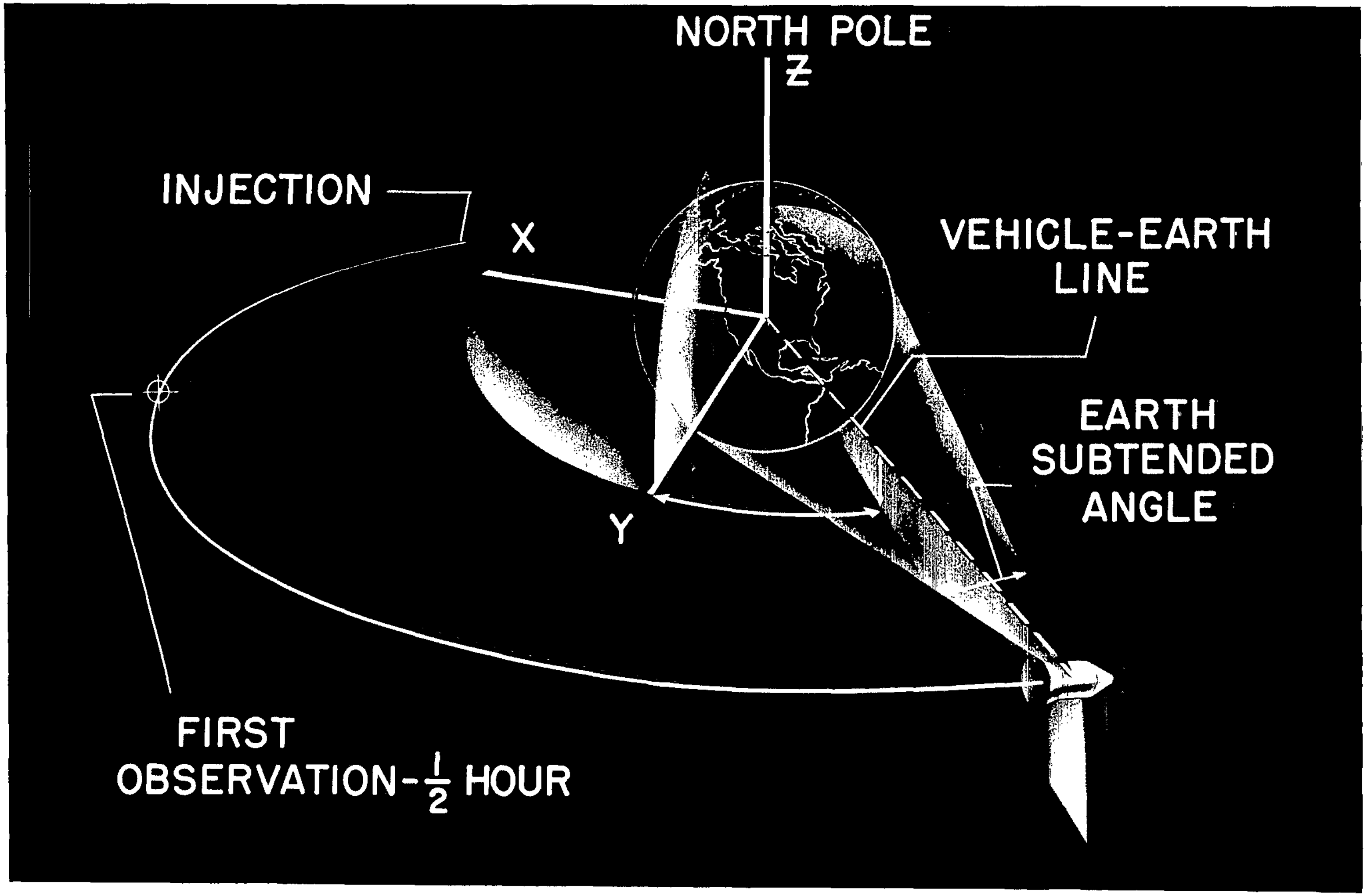

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Figure 4.- Trajectory in the vicinity of the Earth showing observation angles.

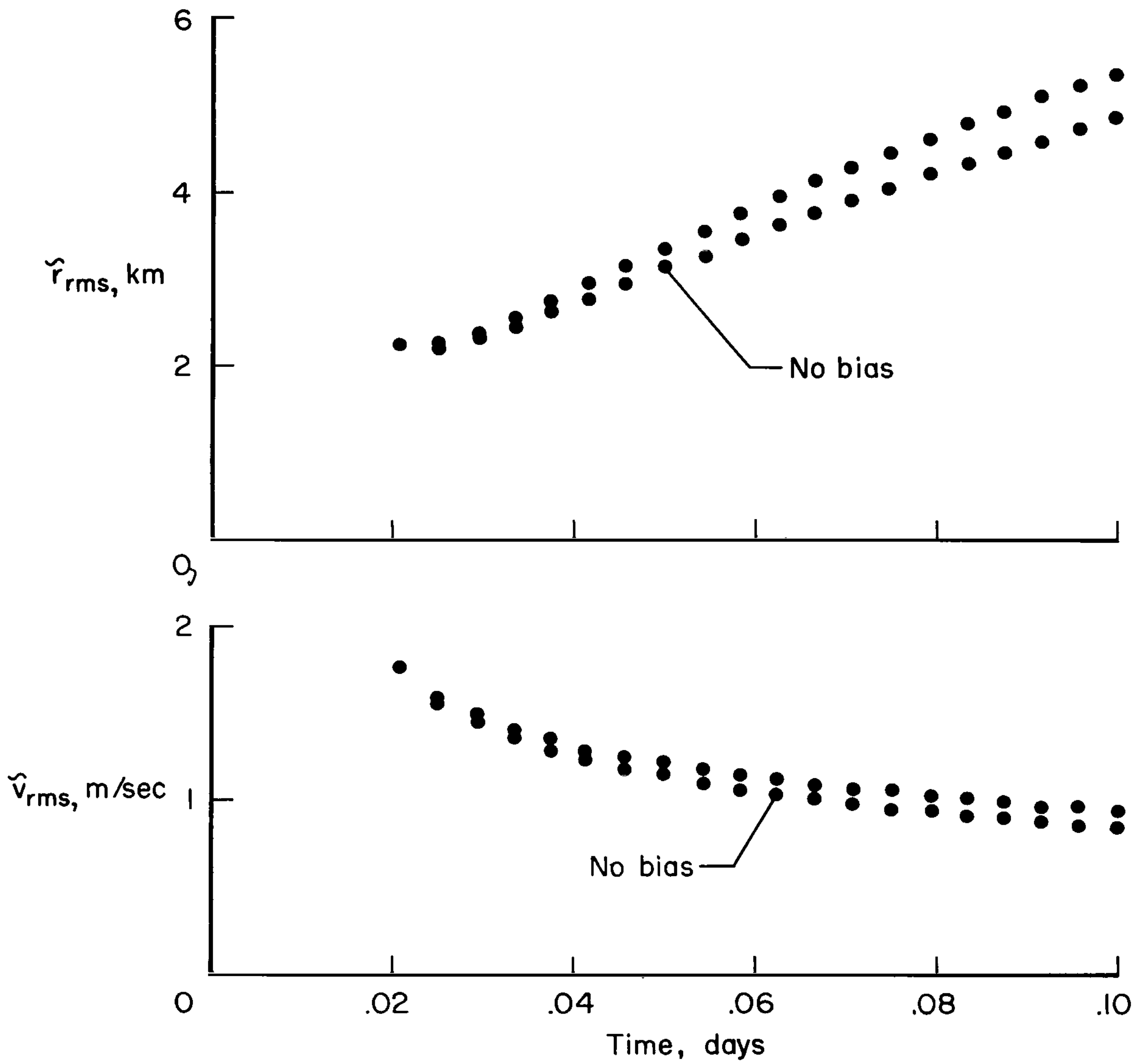

Figure 5.- History of the rms error in estimating position and velocity with and without γ bias, 5 sec arc.

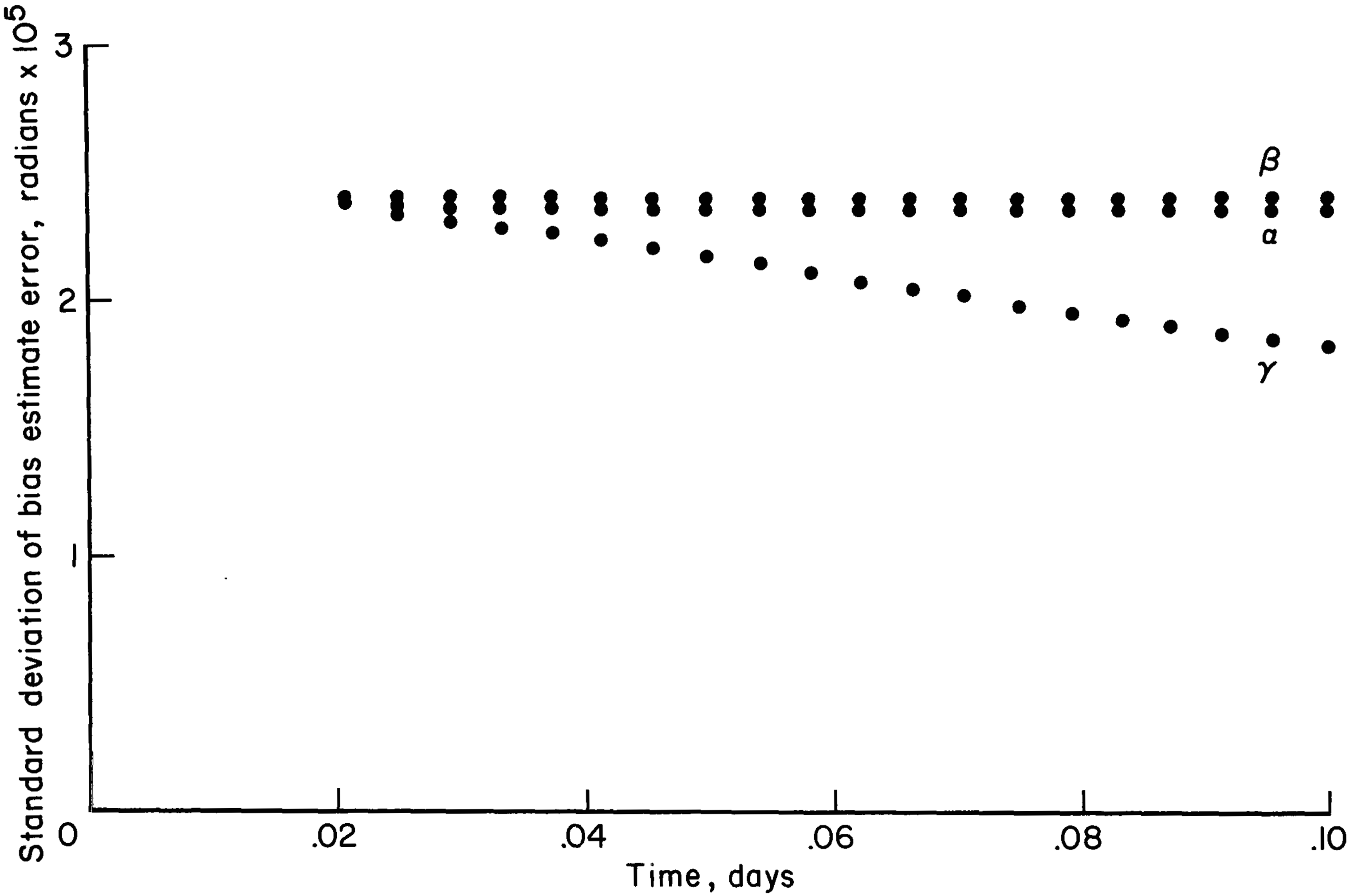

Figure 6.- History of rms bias estimation error.

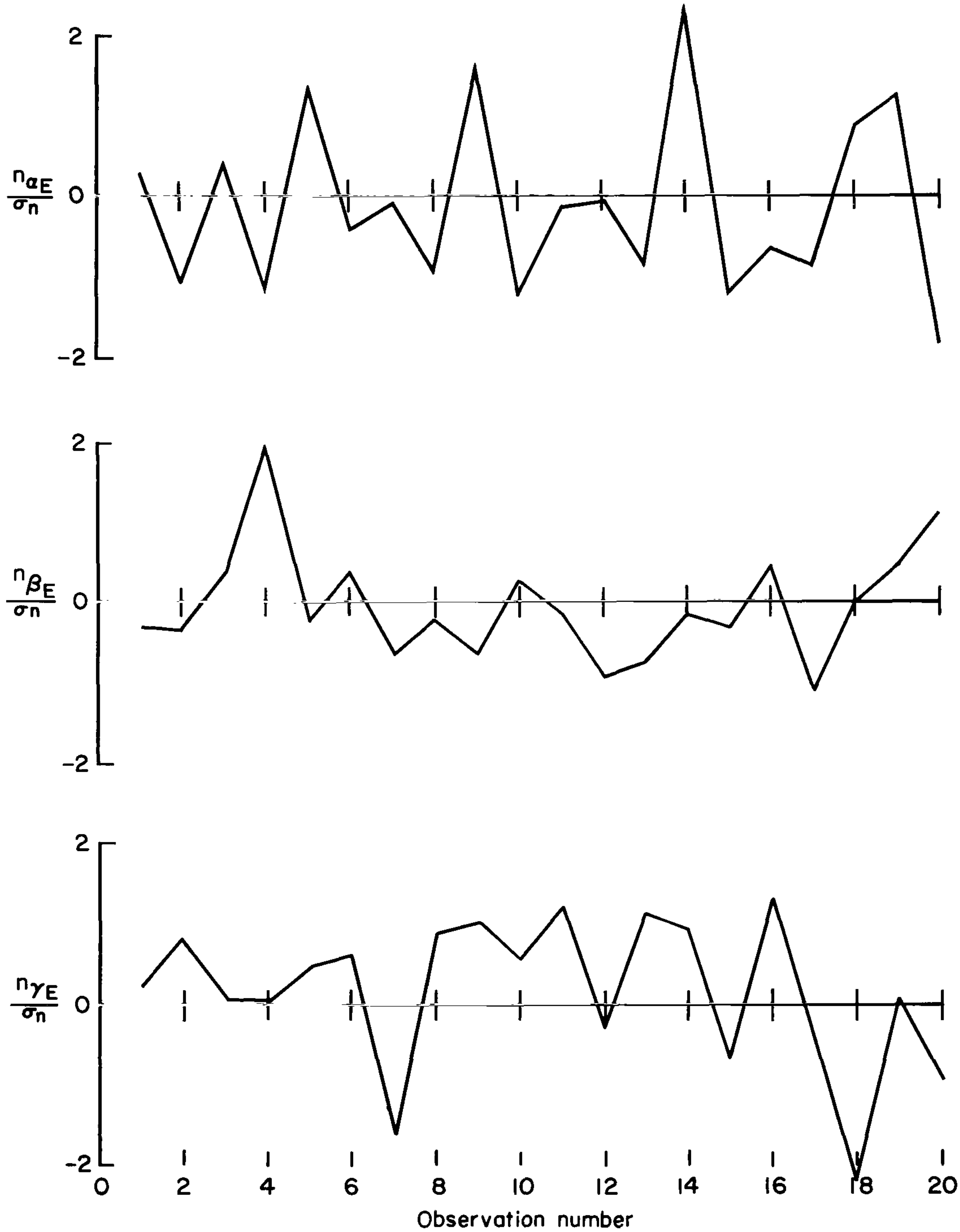

Figure 7.- Time histories of observation errors assumed.

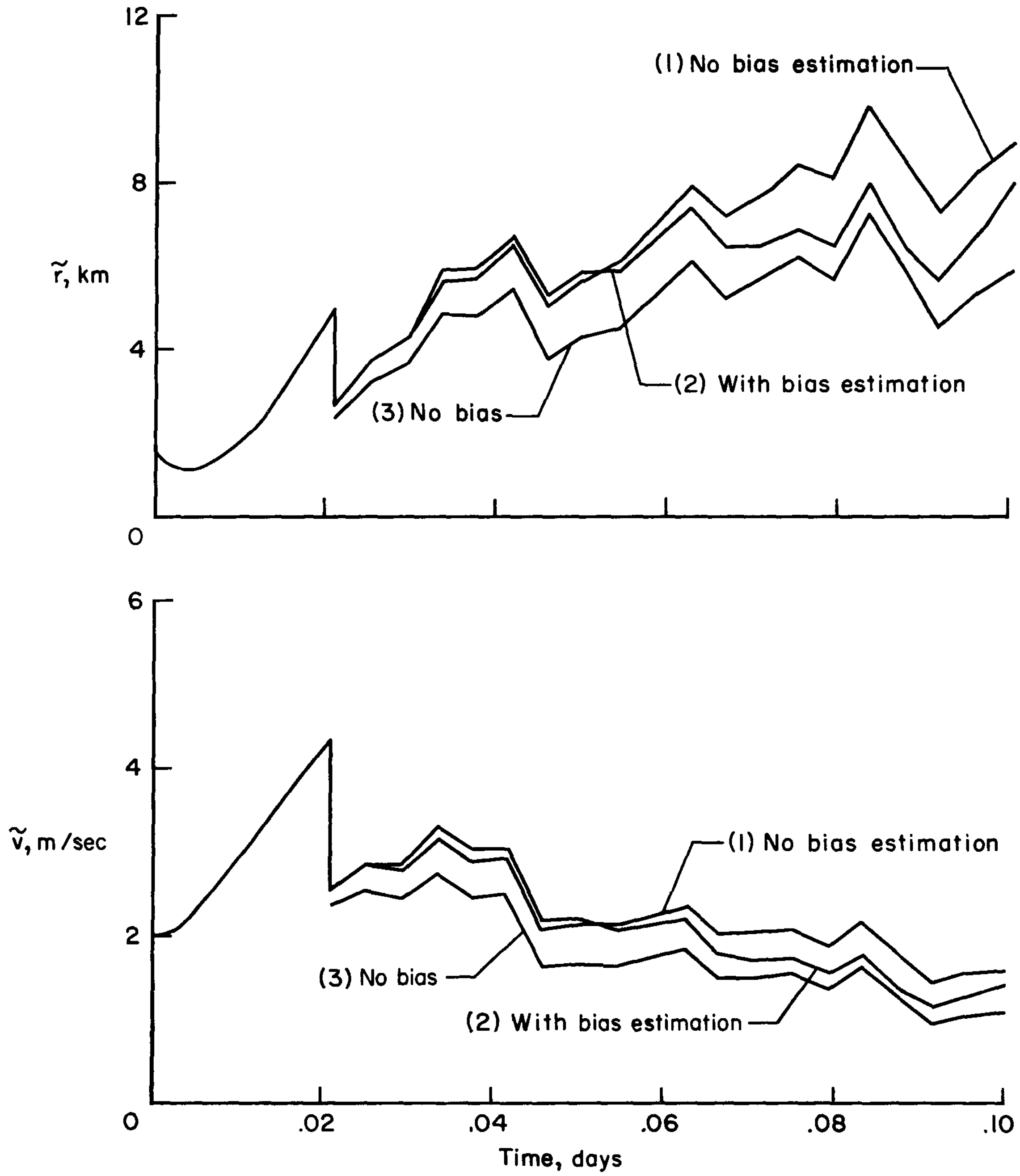

Figure 8.- History of error in estimating position and velocity for a simulated flight, γ bias = 5 sec arc.

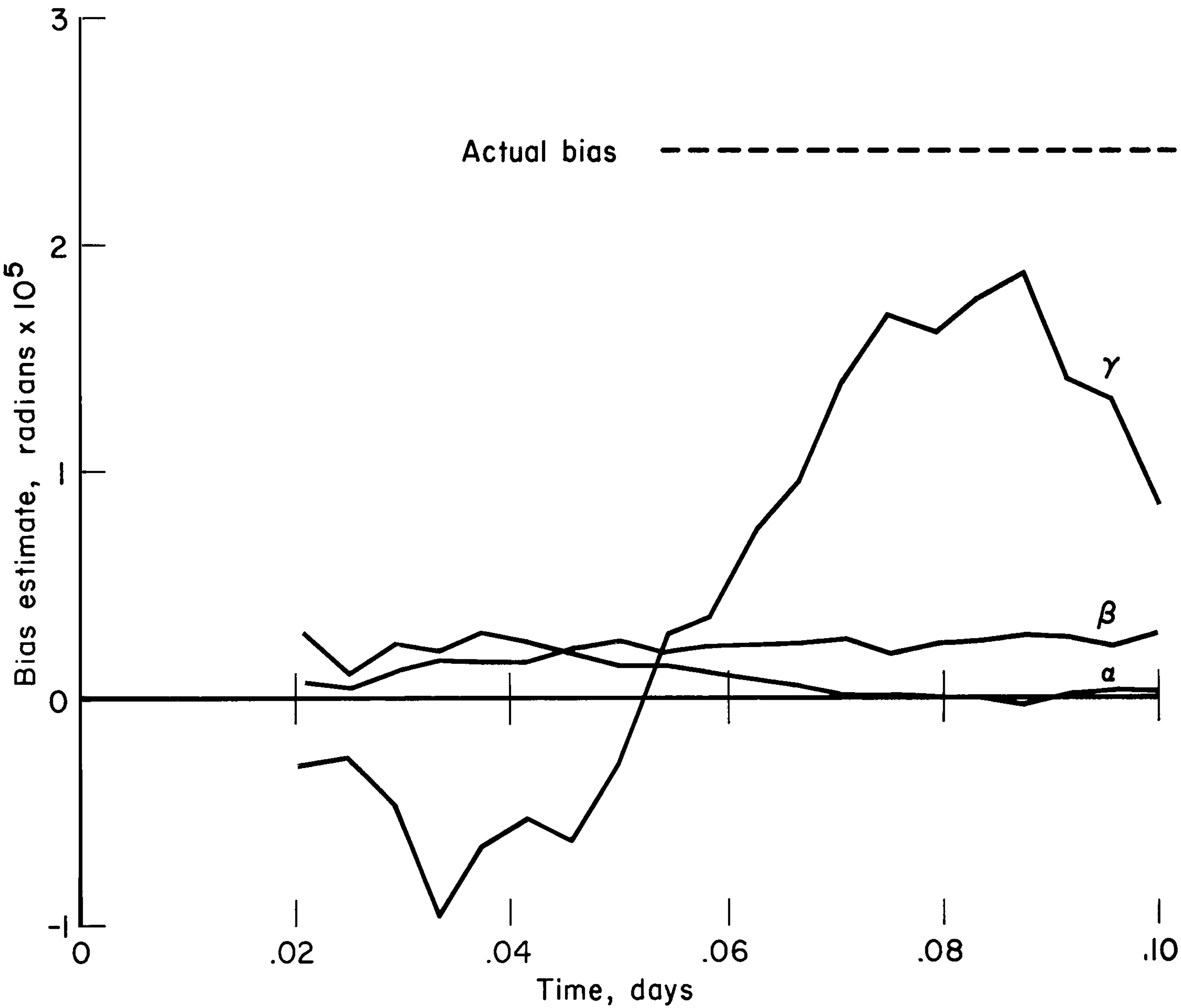

Figure 9.- History of bias estimates for a simulated flight.

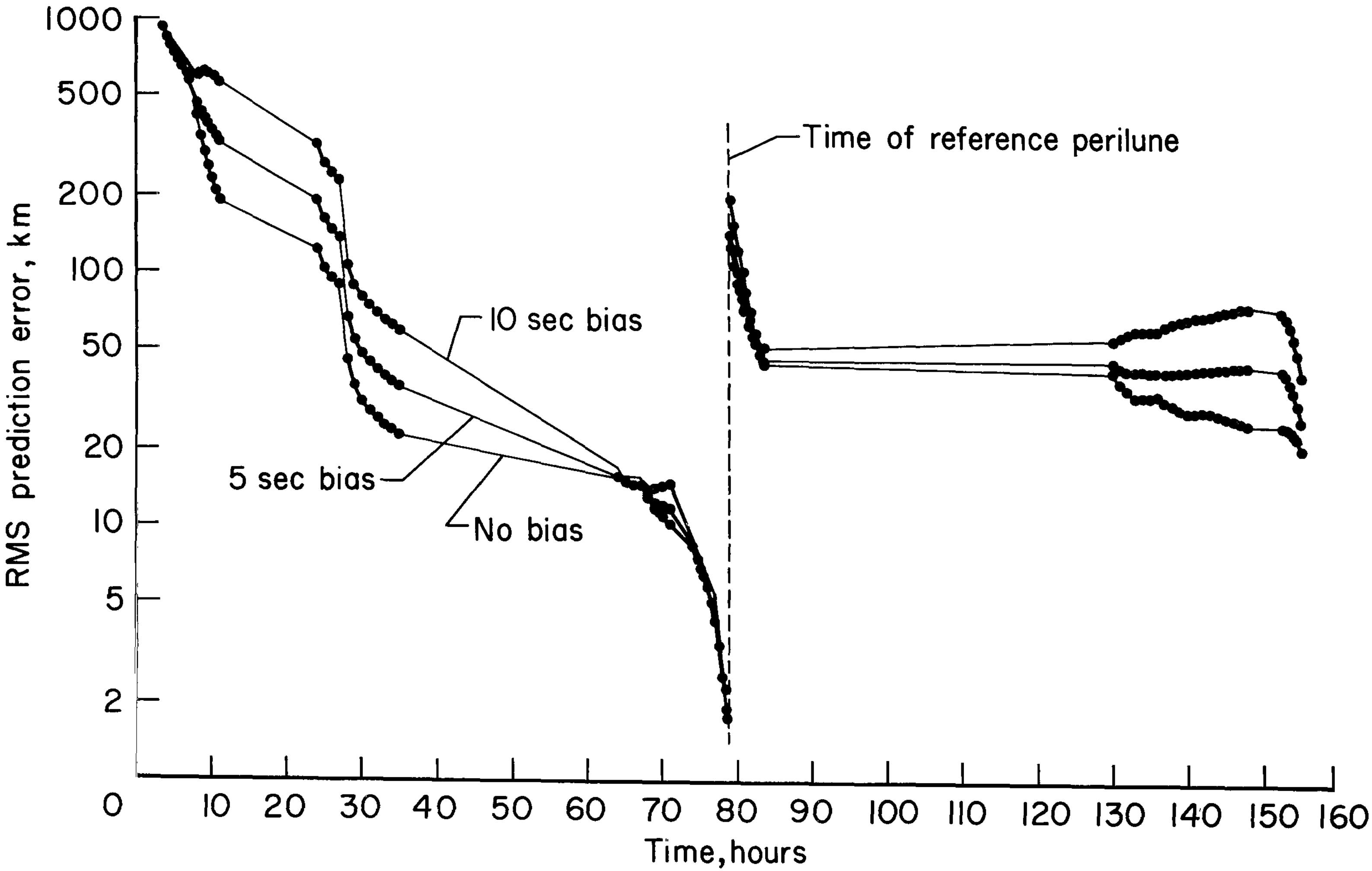

Figure 10.- Performance with uncompensated bias on α, β, γ (80-observation schedule).

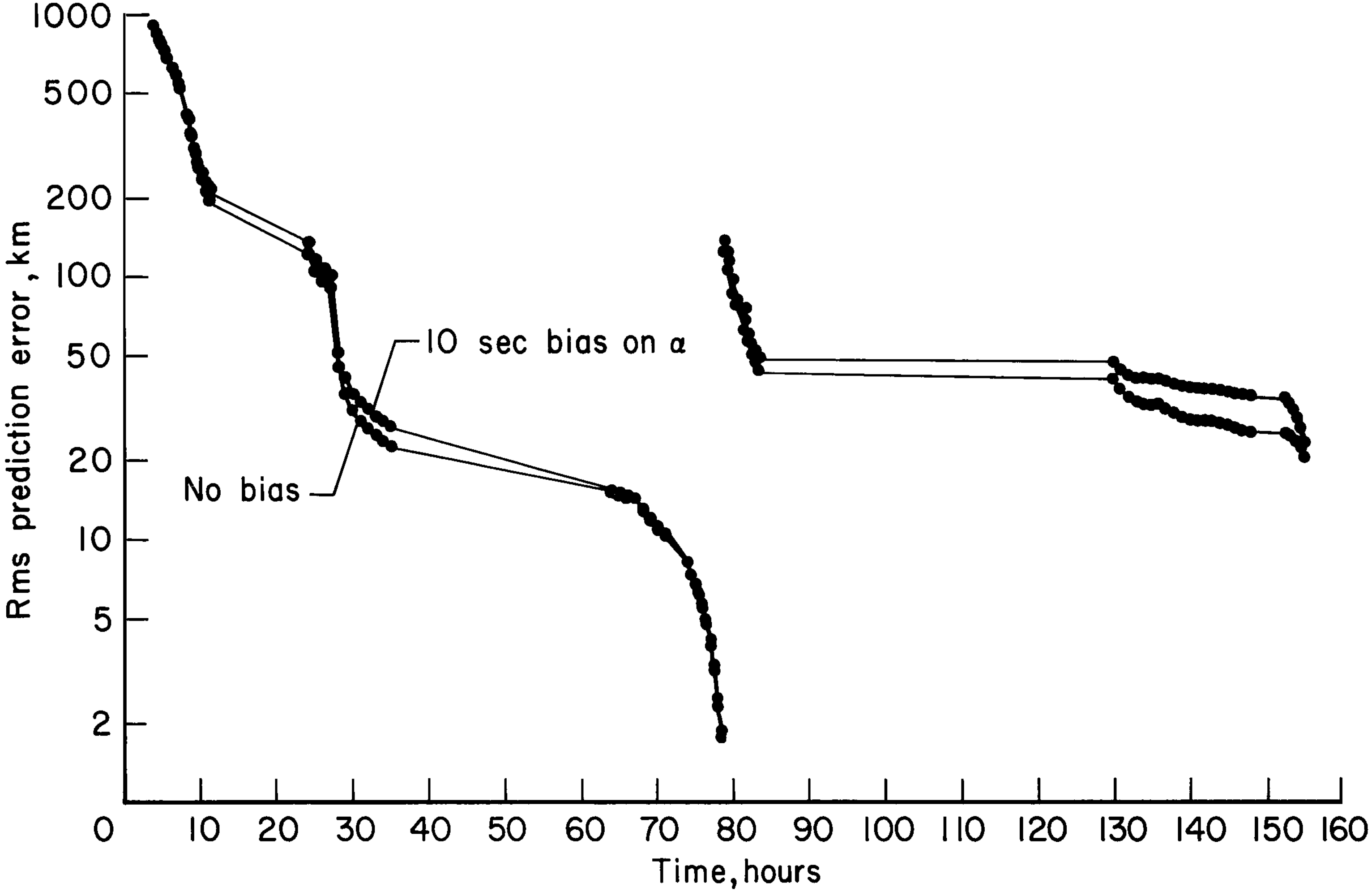

Figure 11.- Performance with uncompensated bias on α (80-observation schedule).

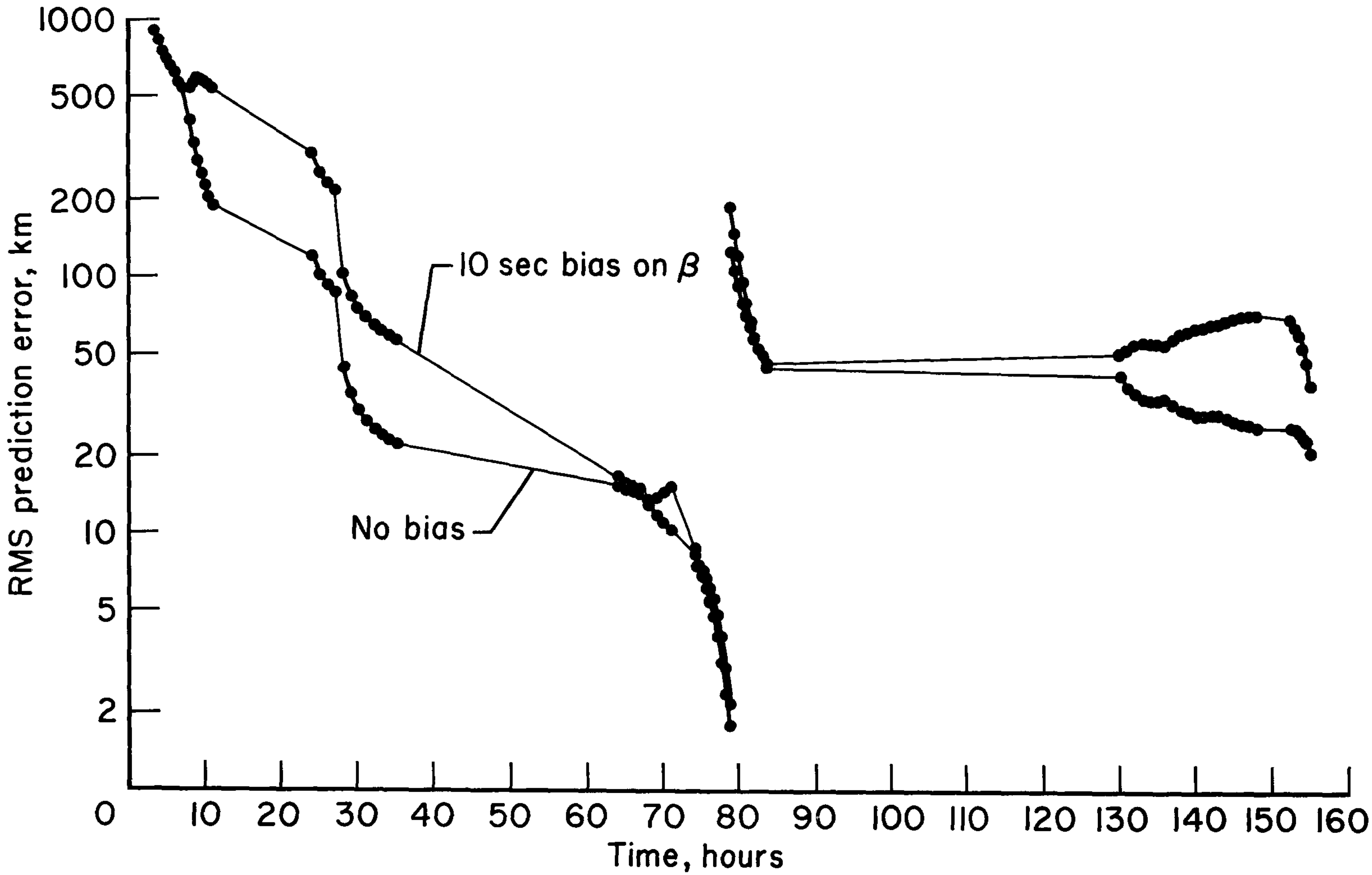

Figure 12.- Performance with uncompensated bias on β (80-observation schedule).

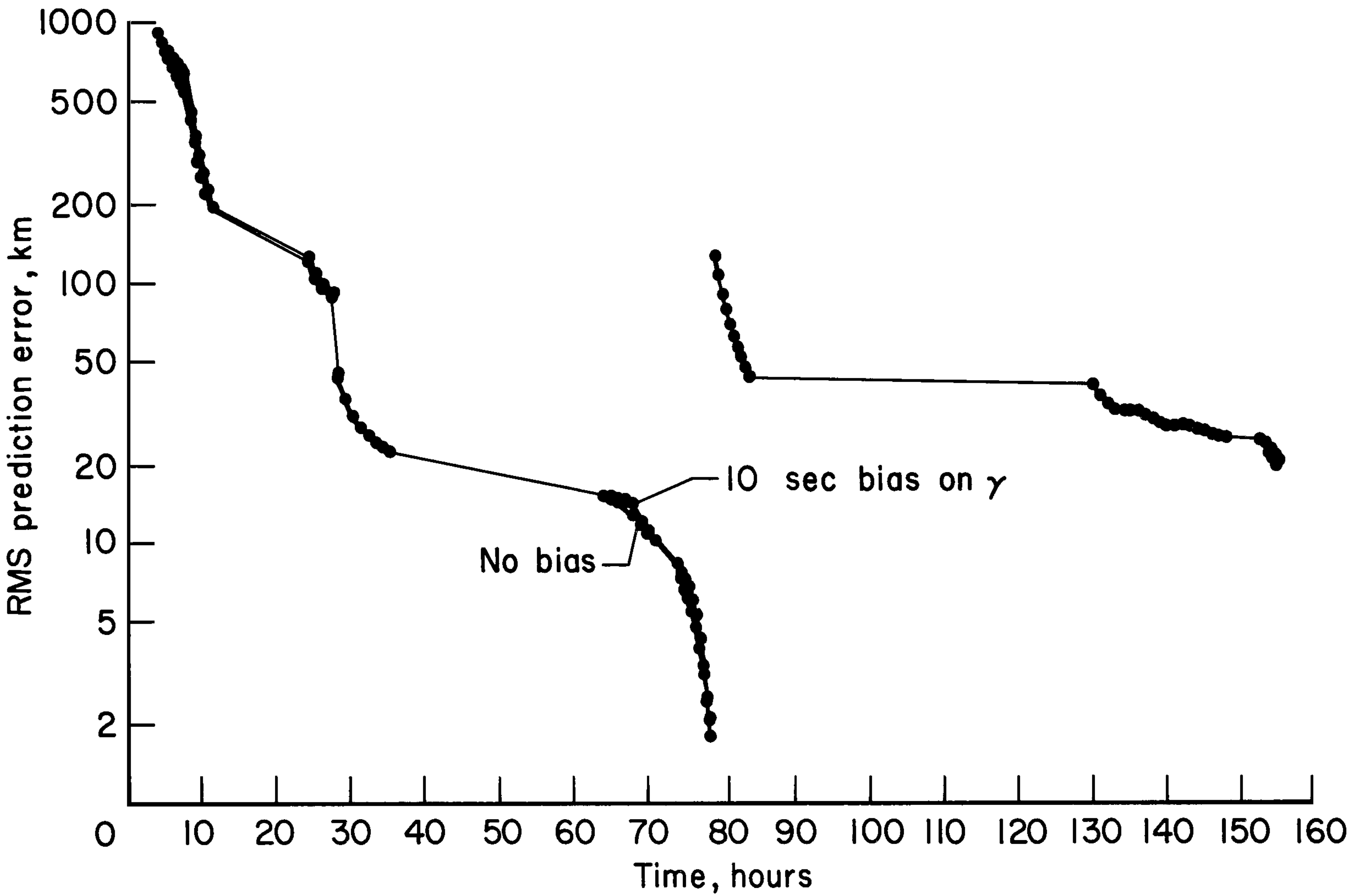

Figure 13.- Performance with uncompensated bias on γ (80-observation schedule).

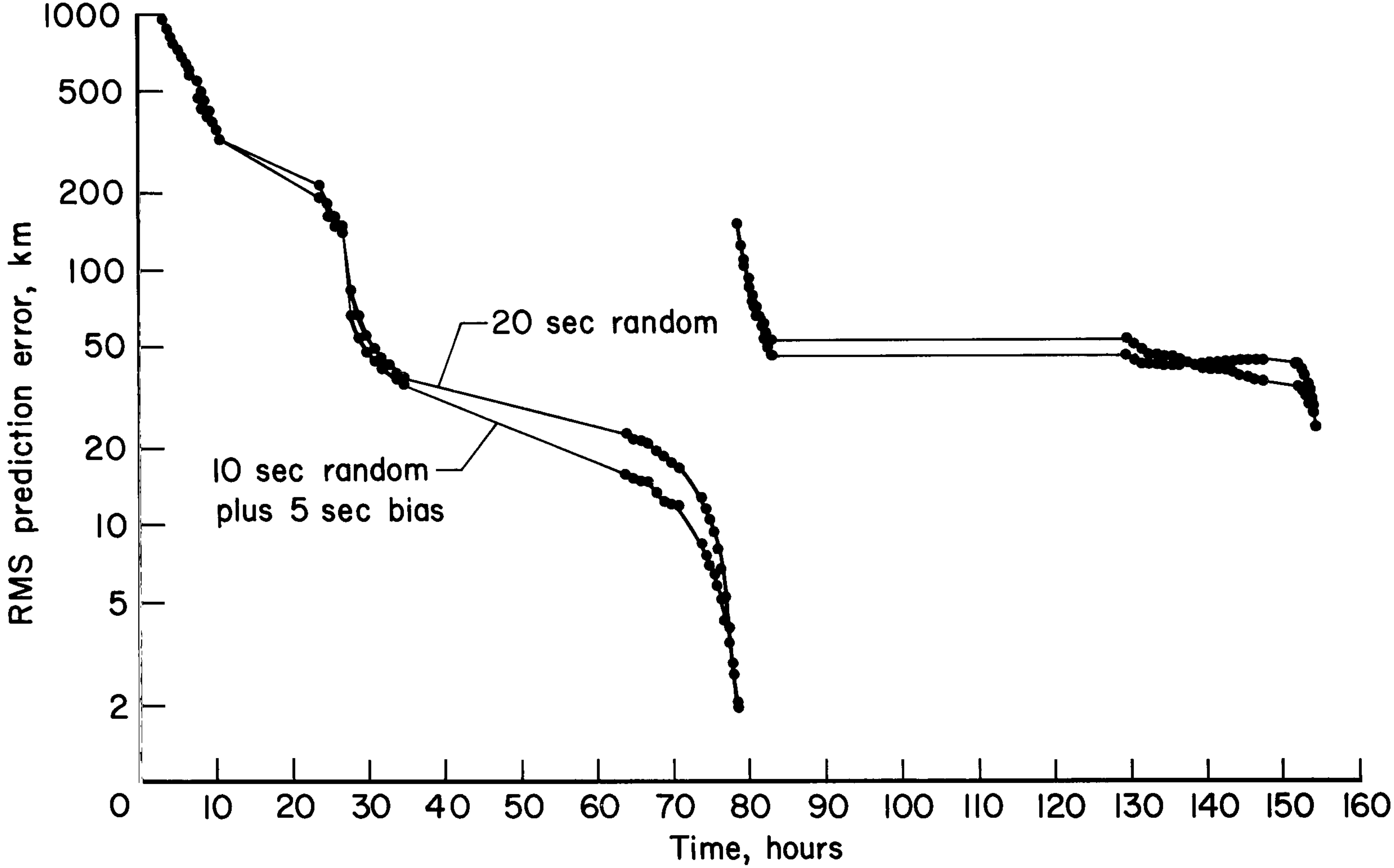

Figure 14.- Equivalence of systems with and without uncompensated bias.

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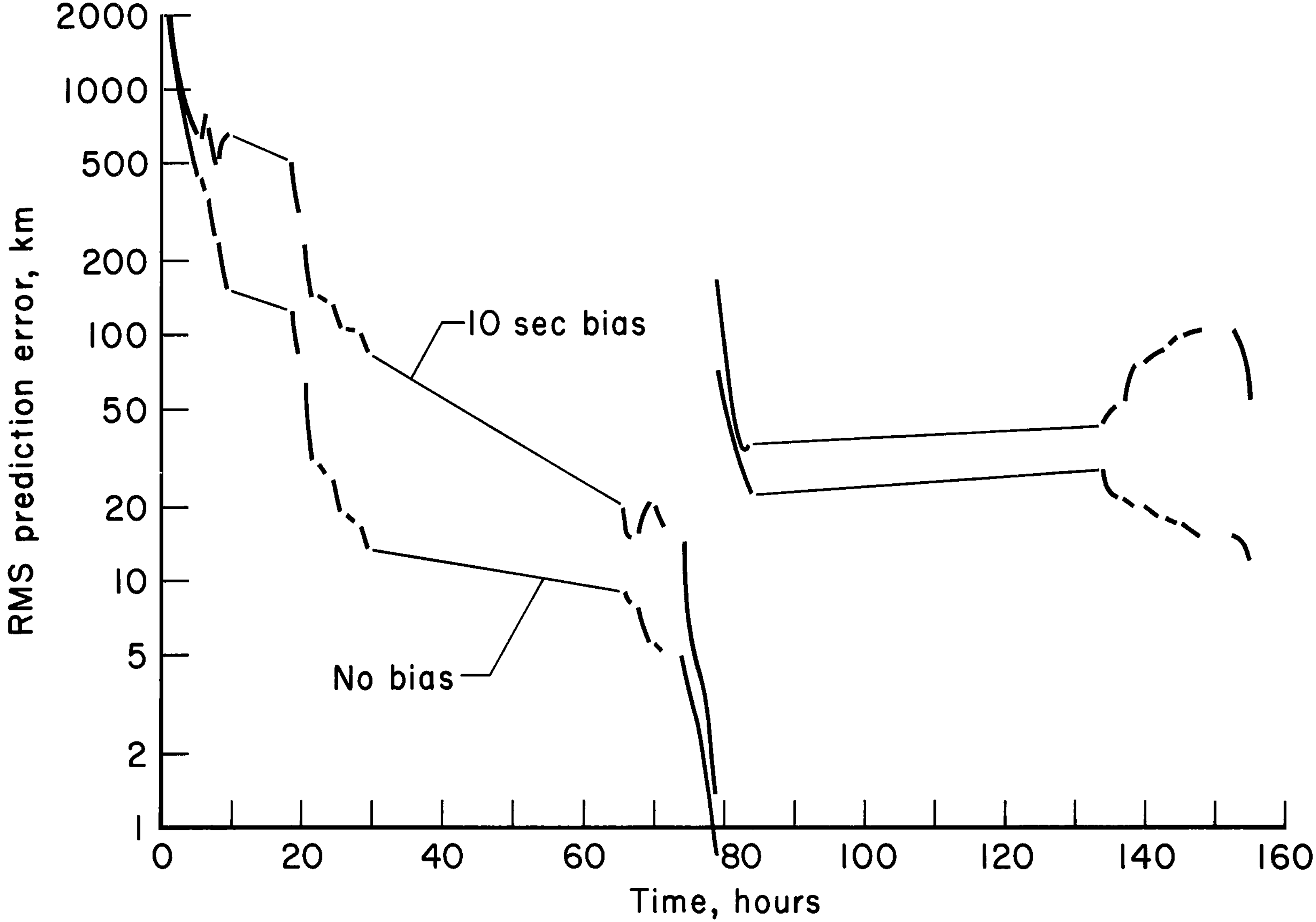

Figure 15.- Performance with uncompensated bias on α, β, γ (400-observation schedule).